Analytic methods for two-loop amplitudes in QCD

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Multi-loop calculations (Methods and applications)
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Science & Technology Facilities Council

IP3

European Research Council
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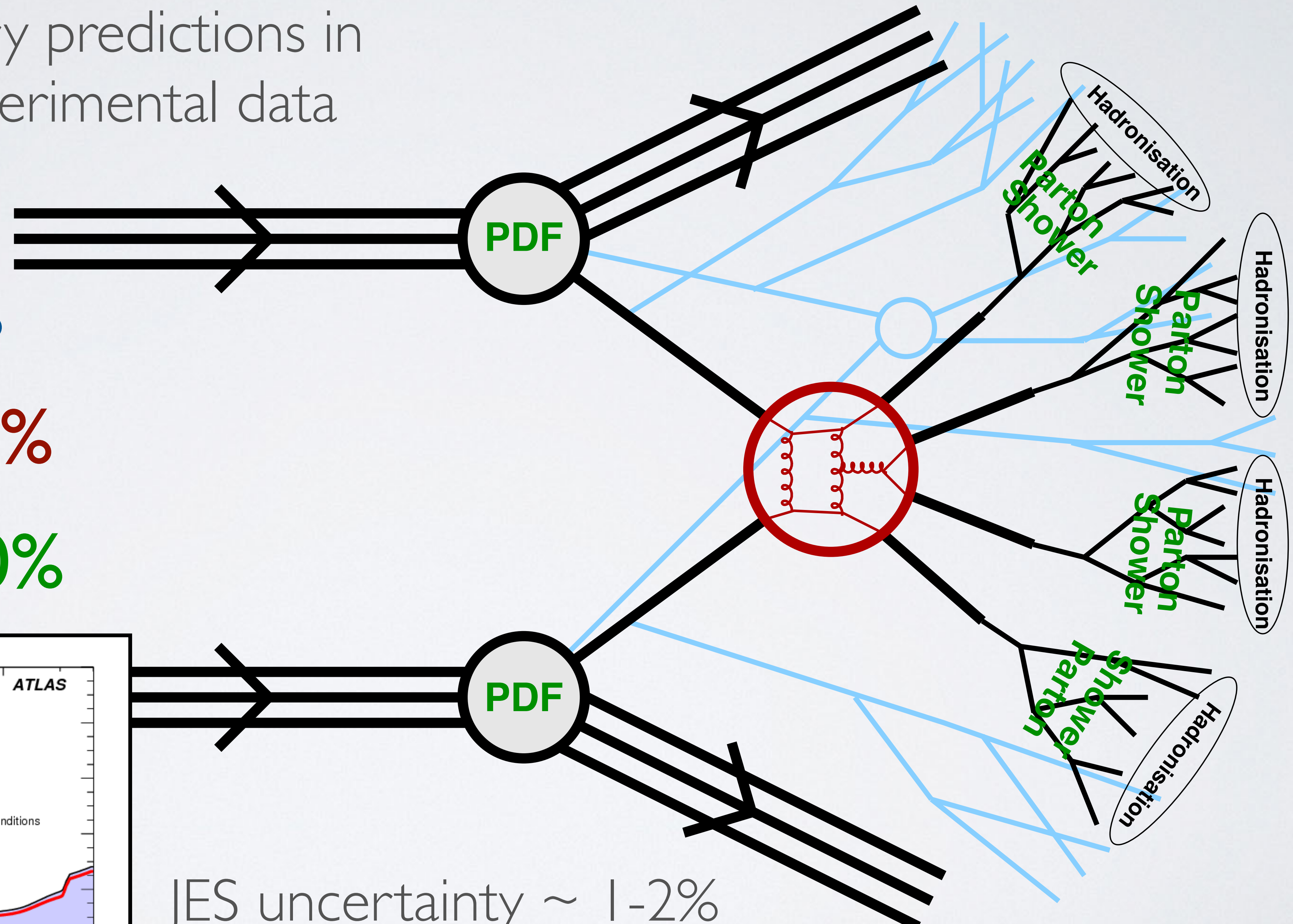
precision hadron collisions

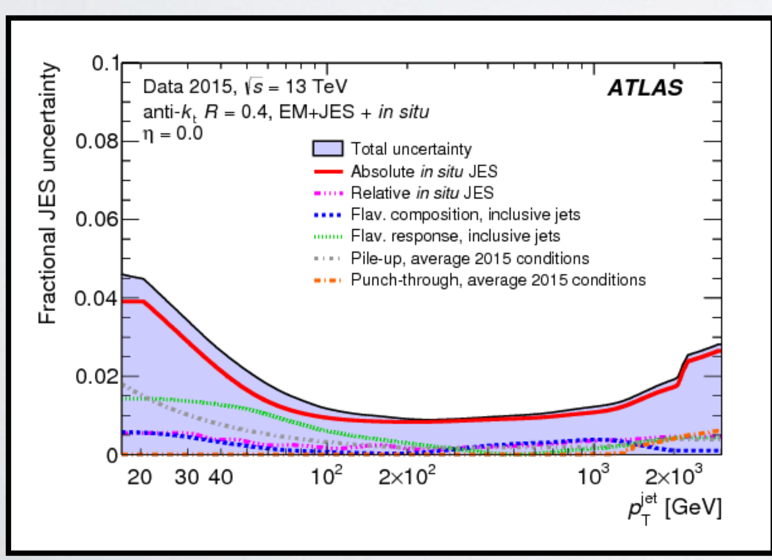
Keeping theory predictions in line with experimental data



LO > 50%

NLO 20-30%

NNLO 5-10%



JES uncertainty ~ 1-2%

$\Rightarrow \sigma$ ~ 5-10%

the NNLO frontier

new subtractions methods $\Rightarrow$ (almost) complete set of 2→2 processes at NNLO!

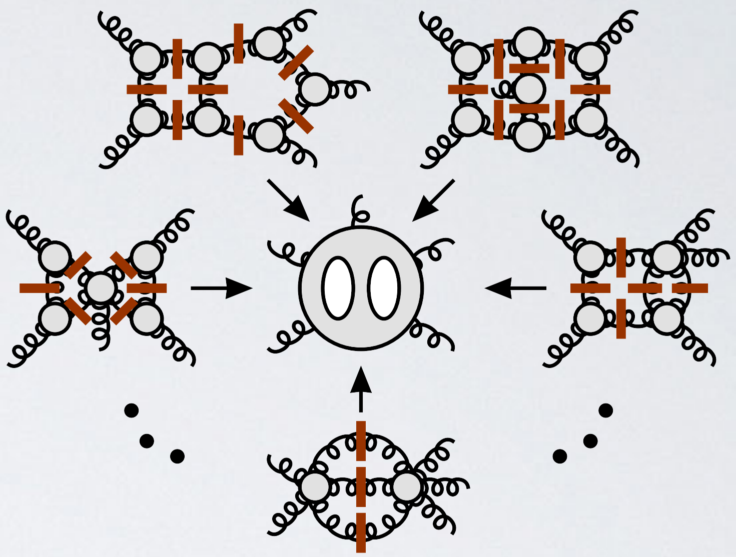
qT, n-jettiness, antenna, sector decomposition/STRIPPER

process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

example: 3j/2j ratio at the LHC can probe of the running of α_s in a new energy regime

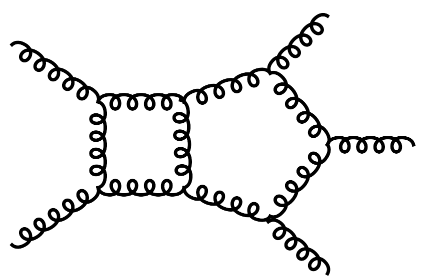
e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\mathrm{exp.}) \pm 0.0018(\mathrm{PDF}) \pm 0.0050(\mathrm{theory})$

outline



- challenges and latest results
- reconstructing analytic amplitudes using numerical evaluation over finite fields
- **new result:** full colour (non-planar) five gluon all-plus amplitude
- future outlook

planar amplitudes in QCD



planar gluon scattering

		$2 \to 2$		$2 \to 3$	
		$\mathcal{N}=4$	QCD	$\mathcal{N}=4$	QCD
one loop	integrand basis	1	65	5	175
	master integrals	1	2	1	2
two loops	integrand basis	2	15360	15	55580
	master integrals	1	7	3	61

new basis of functions

reduction bottlenecks

large algebraic expressions

numerical algebra and an analytic integral basis solved this issue at one-loop

large systems of equations

Laporta-style integration-by-parts systems can easily run into millions for state-of-the-art applications

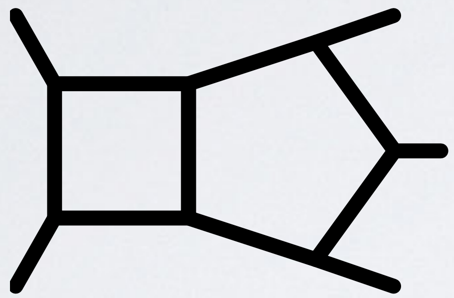
on-shell quantities can be extremely simple - when you find the right language!

e.g. unitarity (Bern, Dixon, Dunbar, Kosower), on-shell recursion (Britto, Cachazo, Feng, Witten)

five-point master integrals

using differential equations

planar



[Papadopoulos, Tommasini, Wever (2015)]

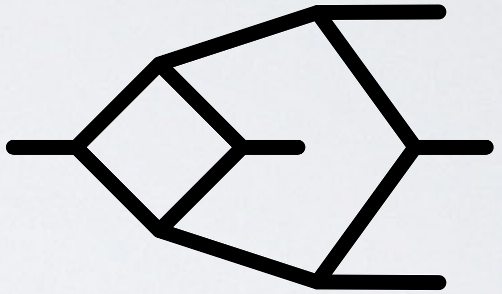
[Gehrmann, Henn, Lo Presti (2015)]

new! [Gehrmann, Henn, Lo Presti 1807.09812]

↓

compact analytic expressions
using 'pentagon function' basis

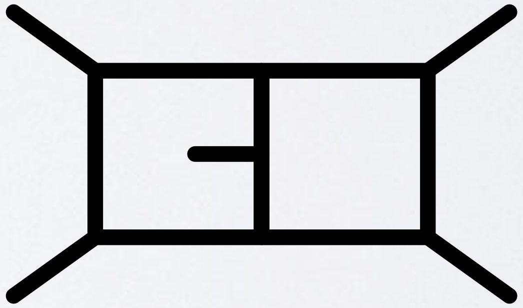
new! non-planar



[Boehm et al. 1805.01873]

[Abreu et al.1807.11522]

[Chicherin et al.1809.06240]



[Chicherin et al. 1812.11057, 1812.11160, 1901.05932]

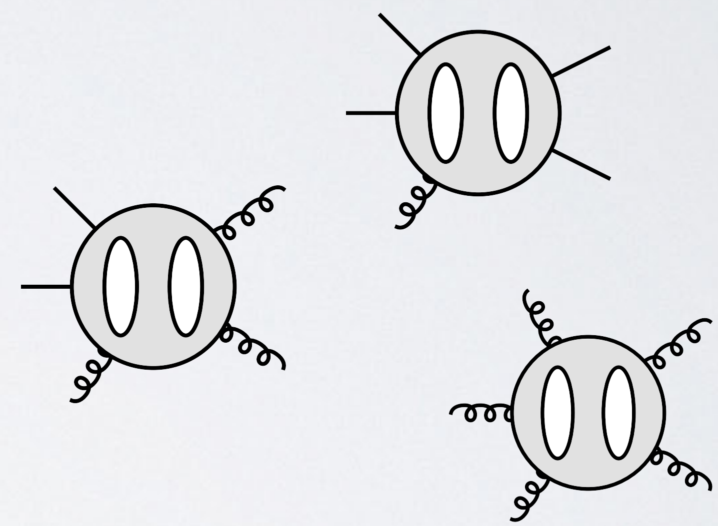
[Abreu et al. 1812.08941,1901.08563]

five-point helicity amplitudes

combining of analytic integrals with analytic reduction to master integrals using finite field reconstruction methods

planar five-gluon single-minus [SB et al. 1811.11699]

planar five-gluon MHV [Abreu et al. 1812.04586]



planar five-parton MHV [Abreu et al. 1904.00945]

non-planar five-gluon N=4 [Abreu et al. 1812.08941, Chicherin et al. 1812.11057]

non-planar five-gluon N=8 [Chicherin et al. 1901.05932, Abreu et al. 1901.08563]

non-planar five-gluon all-plus [SB et al. 1905.03733]

focus for this talk

computational strategy

numerical sampling
with finite field arithmetic

von Mantueffel, Schabinger (2015), Peraro (2016)

unitarity cuts

integrand reduction

integration-by-parts reduction

steps combined with numerical unitarity approach [Gluza, Kadja, Kosower (2011)] [Ita (2015)][Larsen Zhang (2015)][Kosower (2018)]

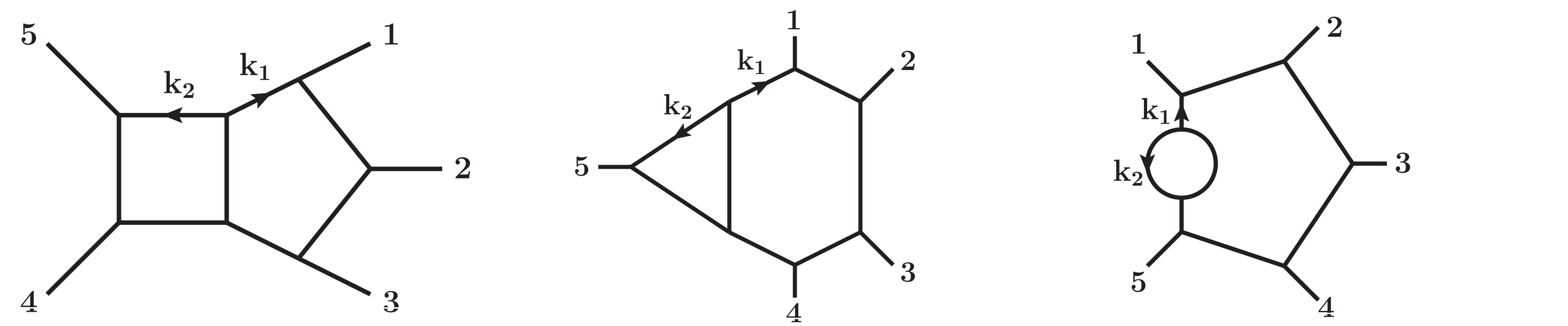
Laurent expansion of master integrals

subtract universal IR poles

analytic reconstruction of finite remainder

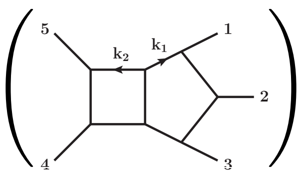
integrand reduction

[Mastrolia, Ossola (2011)][SB, Frellesvig, Zhang (2012)]
[Zhang (2012)][Mastrolia, Mirabella, Ossola, Peraro (2012)]



(a) Pentabox: $G_{11111111a_9a_{10}a_{11}}$ (b) Hexatriangle: $G_{111111a_711a_{10}a_{11}}$ (c) Heptabubble: $G_{21111a_6a_711a_{10}a_{11}}$

integrand basis formed from maximal cut topologies

$\Delta\Bigg($  $\Bigg) = \sum c_{(1,1,1,1,1,1,1,1,a_9,a_{10},a_{11})}(k_1+p_5)^{-2a_9}(k_2+p_1)^{-2a_{10}}(k_2+p_1+p_2)^{-2a_{11}}$

complete integrand is not reconstructed -
numerical solution passed to next step (IBPs)

integration-by-parts reduction

Laporta solution to system of IBP equations using finite field sampling

modular implementation into FiniteFlow C++ code (Peraro)

$\boxed{\text{complete system is not reconstructed - numerical solution passed to next step}}$

analytic integrals and IR poles

$$A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{ finite}$$

universal **ε** pole structure
e.g. [Catani (1998)], [Sterman, Tejeda-Yeomans (2003)]
[Becher, Neubert (2009)], [Gardi, Magnea (2009)]

higher order terms in **ε** needed

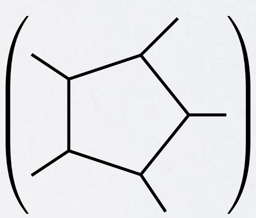
$$MI_x = \sum_{y,z} \epsilon^y c_{xyz} m_{xyz}(f(W))$$

alphabet of 31 letters

monomials of **pentagon function** integrals. Li_n etc. and one-fold integrals

[Gehrmann, Henn, Lo Presti (2018)]

$I^{[6-2\epsilon]}$



$\propto f_{3,4}$

$$f_{3,4} - \frac{2}{3} d_{37,3} = \int_0^1 dt\, \partial_t \log W_{26}(t) \left(\left[\frac{W_3}{W_5}, \frac{W_2}{W_{15}}\right](t) - \left[\frac{W_5}{W_2}, \frac{W_3}{W_{12}}\right](t) - \zeta_2 \right) + \text{cyclic}$$

analytic integrals and IR poles

$$A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{ finite}$$

direct construction of finite remainder from (6d) tree-level amplitudes via **unitarity cuts, integrand reduction, IBP reductions** and reconstruction of rational coefficients of the **pentagon function basis**

- performance depends on the complexity (polynomial order) of the rational functions in the final answer
- rational parametrisation of external kinematics from momentum twistors
- extremely effective when cancellations are explicit
- numerical sampling is easily parallelised

Applications

Analytic helicity amplitudes for two-loop five-gluon scattering: the single-minus case

[SB, Brønnum-Hansen, Hartanto, Peraro]

Analytic form of the full two-loop five-gluon all-plus helicity amplitude

[SB, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia]

single-minus finite remainders

SB, Brønnum-Hansen, Hartanto, Peraro [1811.11699]

exponent of d_s-2
where $d_s=g^{\mu}{}_{\mu}$

$$F^{(L),[i]}\left(1_g^-, 2_g^+, 3_g^+, 4_g^+, 5_g^+\right) = \frac{[25]^2}{[12]\langle 23\rangle\langle 34\rangle\langle 45\rangle[51]}\left(F_{\mathrm{sym}}^{(L),[i]}(1,2,3,4,5) + F_{\mathrm{sym}}^{(L),[i]}(1,5,4,3,2)\right)$$

leading order

$$F_{\mathrm{sym}}^{(1),[1]}(1,2,3,4,5) = \frac{\mathrm{tr}_+(2315)^2\,\mathrm{tr}_+(1243)}{3s_{25}^2 s_{23}s_{34}s_{15}} - \frac{\mathrm{tr}_+(2543)}{6s_{34}}$$

finite at one-loop:
compact expression [Bern Dixon Kosower (1993)]

single-minus finite remainders

- large cancellation between two-loop amplitude and universal poles
- only weight two functions remain

$d_s - 2$ exponent	0	1	2
integrand coefficients ($\neq 0$)	4387	14565	4420
sample points	3	2214	22886

one-loop squared
- very fast

in this example - the run time is fast and additional optimisation was unnecessary

coefficients are found to be even simpler than this sampling suggests…

single-minus finite remainders

$$
\begin{aligned}
F^{(2),[1]}_{\mathrm{sym}}(1,2,3,4,5) = \; & c^{(2)}_{51} F^{(2)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(1)}_{51} F^{(1)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(0)}_{51} F^{(0)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) \\
& + c^{(2)}_{34} F^{(2)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(1)}_{34} F^{(1)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(0)}_{34} F^{(0)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) \\
& + c_{45} F^{(0)}_{\mathrm{box}}(s_{12},s_{23},s_{45}) + c_{34;51} \hat{L}_1(s_{34},s_{15}) + c_{51;23} \hat{L}_1(s_{15},s_{23}) + c_{\mathrm{rat}},
\end{aligned}
$$

compact 2-loop expressions - free of spurious singularities using one-loop-type (weight 2) function basis (c.f. Bern, Dixon, Kosower (1993,1994))

$$
L_k(s,t) = \frac{\log(t/s)}{(s-t)^k},
$$

$$
\begin{aligned}
\hat{L}_0(s,t) &= L_0(s,t), \\
\hat{L}_1(s,t) &= L_1(s,t), \\
\hat{L}_2(s,t) &= L_2(s,t) + \frac{1}{2(s-t)}\left(\frac{1}{s}+\frac{1}{t}\right), \\
\hat{L}_3(s,t) &= L_3(s,t) + \frac{1}{2(s-t)^2}\left(\frac{1}{s}+\frac{1}{t}\right).
\end{aligned}
$$

$$
F^{(-1)}_{\mathrm{box}}(s,t,m^2) = \mathrm{Li}_2\left(1-\frac{s}{m^2}\right) + \mathrm{Li}_2\left(1-\frac{t}{m^2}\right) + \log\left(\frac{s}{m^2}\right) + \log\left(\frac{t}{m^2}\right) - \frac{\pi^2}{6}, \tag{4.7a}
$$

$$
F^{(0)}_{\mathrm{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)} F^{(-1)}_{\mathrm{box}}(s,t,m^2), \tag{4.7b}
$$

$$
F^{(1)}_{\mathrm{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(0)}_{\mathrm{box}}(s,t,m^2) + \hat{L}_1(s,m^2) + \hat{L}_1(m^2,t)\right], \tag{4.7c}
$$

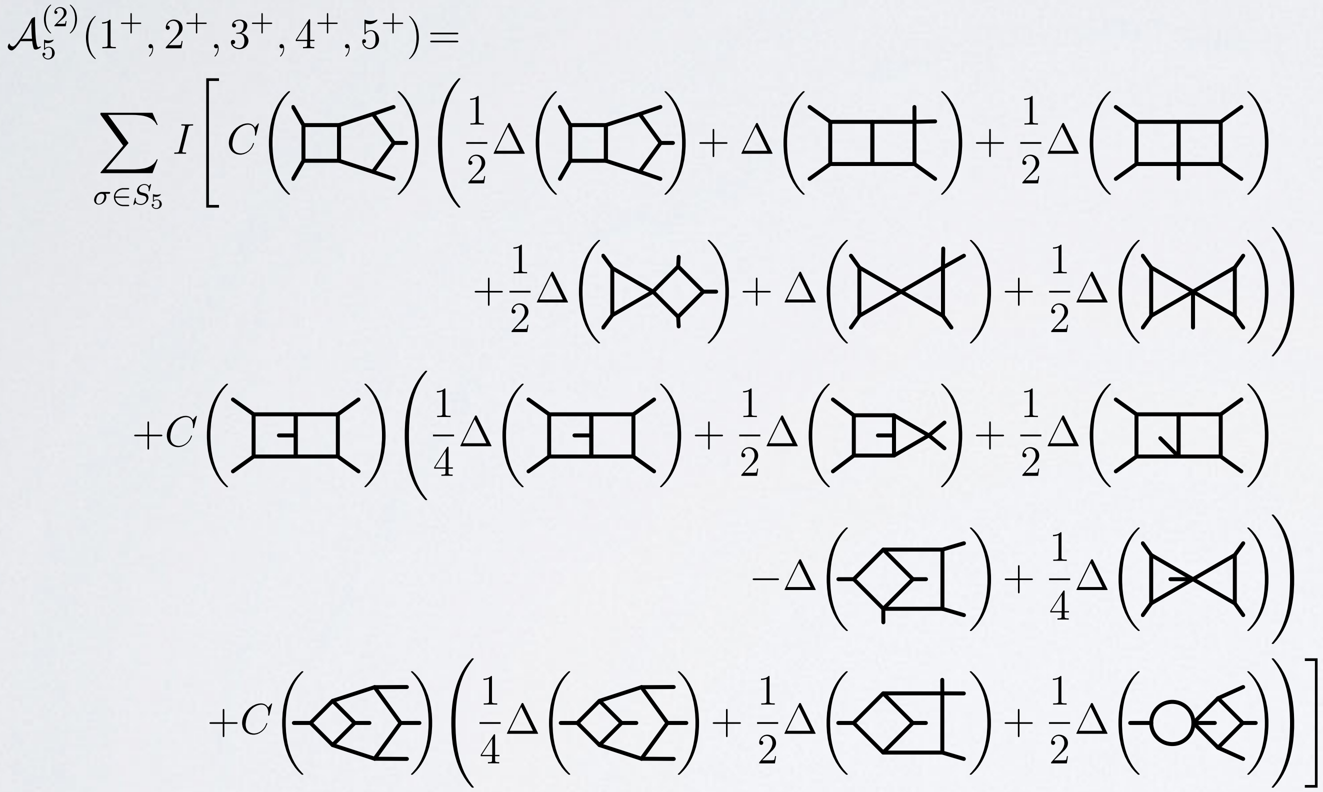
$$
\begin{aligned}
F^{(2)}_{\mathrm{box}}(s,t,m^2) = & \frac{1}{u(s,t,m^2)}\Bigg[F^{(1)}_{\mathrm{box}}(s,t,m^2) + \frac{s-m^2}{2t}\hat{L}_2(s,m^2) + \frac{m^2-t}{2s}\hat{L}_2(m^2,t) \\
& - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{4m^2}\Bigg],
\end{aligned} \tag{4.7d}
$$

$$
\begin{aligned}
F^{(3)}_{\mathrm{box}}(s,t,m^2) = & \frac{1}{u(s,t,m^2)}\Bigg[F^{(2)}_{\mathrm{box}}(s,t,m^2) - \frac{(s-m^2)^2}{6t^2}\hat{L}_3(s,m^2) - \frac{(m^2-t)^2}{6s^2}\hat{L}_3(m^2,t) \\
& - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{6m^4}\Bigg],
\end{aligned} \tag{4.7e}
$$

simple rational coefficients using s_{ij}, tr_+

analytic form of the full two-loop five-gluon all-plus helicity amplitude

SB, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia [1905.03733]



compact analytic integrand [SB, Mogull, Ochirov, O'Connell (2015)]

new: combined with new analytic master integrals and analytic reconstruction of the finite remainder

colour decomposition

integrand constructed using 'multi-peripheral' basis*
and exploiting BCJ† relations between unitarity cuts

for more details see
[Ochirov, Page (2016)]

the integrated result has a compact
representation in the trace basis

$$\mathcal{A}_5^{(1)} = \sum_{\lambda=1}^{12} N_c\, A_\lambda^{(1,0)} \mathcal{T}_\lambda + \sum_{\lambda=13}^{22} A_\lambda^{(1,1)} \mathcal{T}_\lambda \,,$$
$$\mathcal{A}_5^{(2)} = \sum_{\lambda=1}^{12} \left(N_c^2 A_\lambda^{(2,0)} + A_\lambda^{(2,2)} \right) \mathcal{T}_\lambda + \sum_{\lambda=13}^{22} N_c A_\lambda^{(2,1)} \mathcal{T}_\lambda$$

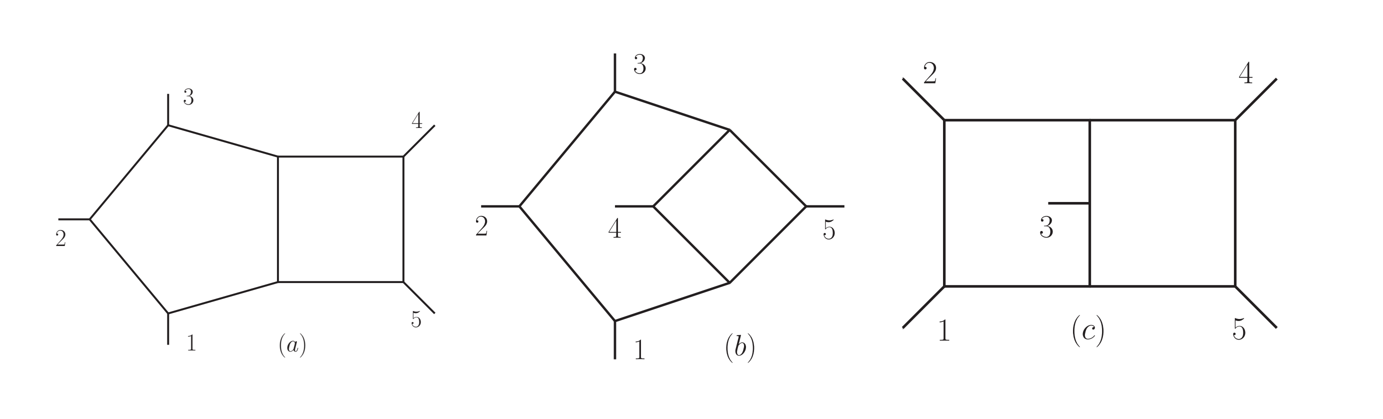
independent partial amplitude

$$\mathcal{T}_1 = \mathrm{Tr}(12345) - \mathrm{Tr}(15432)\,,$$
$$\mathcal{T}_{13} = \mathrm{Tr}(12)\left[\mathrm{Tr}(345) - \mathrm{Tr}(543)\right]$$

* [Del Duca, Dixon, Maltoni (2000)]

†[Bern, Carrasco, Johansson (2008)]

master integral evaluation



evaluation of all master integrals in the region

$$s_{12} > 0\,, \qquad s_{34} > 0\,, \qquad s_{35} > 0\,, \qquad s_{45} > 0 \qquad\qquad s_{1j} < 0\,,\; s_{2j} < 0\,, \quad \text{for } j = 3,4,5$$

analytic evaluation of boundary values

$$X = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{15}\} \qquad X_0 = \{3, -1, 1, 1, -1\}$$

verified with numerically with pySecDec

finite remainders

- cancellation of all weight 1,3 and 4 functions
- verified in all collinear limits
- weight 2 described by box function

$$A_1^{(1,0)} = \frac{\kappa}{5} \sum_{S_{\mathcal{T}_1}} \left[\frac{[24]^2}{\langle 13\rangle\langle 35\rangle\langle 51\rangle} + 2\,\frac{[23]^2}{\langle 14\rangle\langle 45\rangle\langle 51\rangle} \right] \qquad \kappa = (D_s-2)/6$$

$$\mathcal{H}_1^{(2,0)} = \sum_{S_{\mathcal{T}_1}} \left\{ -\kappa\,\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\, I_{123;45} + \kappa^2 \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},$$

$$\mathcal{H}_{13}^{(2,1)} = \sum_{S_{\mathcal{T}_{13}}} \left\{ \kappa\,\frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4\, I_{345;12} - 4\, I_{354;12} - 4\, I_{435;12} \right] \right.$$

$$\left. - 6\,\kappa^2 \left[\frac{s_{23}\,\mathrm{tr}_-(1345)}{s_{34}\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - \frac{3}{2}\,\frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle} \right] \right\},$$

conformally invariant building block

outlook

- numerical algorithms and finite field arithmetic can provide powerful tools for analytic amplitude computations
- traditional bottlenecks can be avoided by direct reconstruction of on-shell physical quantities (e.g. finite remainders)
- surge of progress on two-loop amplitudes! major challenge to combine amplitudes into physical cross sections at NNLO