# Fishnet graphs, their continuum limit and their deformations

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LPENS

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Jussieu, 14-15 May

Based on work with De-liang Zhong

# Plan

Conformal fishnet theory in $d = 4$

Continuum limit: fishnet graph and AdS non-linear sigma model

Deformations and conclusions

# Fishnet CFT

A theory for matrix scalar fields with quartic coupling

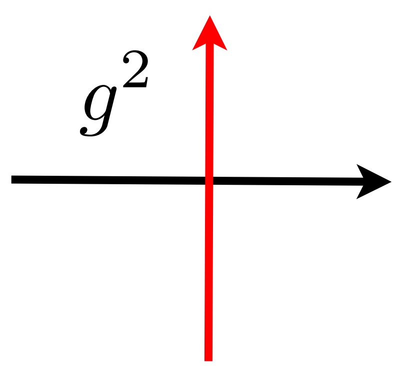
[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]

$$\mathcal{L}_{\text{fishnet}} = N \text{tr} \left[ \partial_\mu \phi_1 \partial_\mu \phi_1^* + \partial_\mu \phi_2 \partial_\mu \phi_2^* + (4\pi g)^2 \phi_1 \phi_2 \phi_1^* \phi_2^* \right]$$

Follows from N = 4 SYM

[Leigh,Strassler'95]
[Frolov'05]
[Lunin,Maldacena'05]
[Beisert,Roiban'05]

a) Twist N=4 SYM theory ($\gamma$ deformation)
b) Double scaling limit ($\gamma \to i\infty$ & $\lambda \to 0$ )




1. **No SUSY**: gluons & gauginos are gone
2. Gauge group becomes a flavour group
3. **Conformal symmetry** is preserved for any coupling in planar limit (caveat: double-trace couplings must be added and tuned)
4. **Integrability** remains

[Grabner,Gromov,Kazakov,Korchemsky'17]
[Sieg,Wilhelm'16]

# Fishnet CFT

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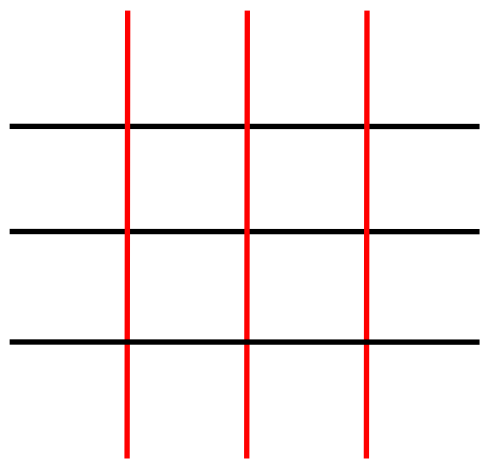
$$\mathcal{L}_{\text{fishnet}} = N\text{tr}\left[\partial_\mu\phi_1\partial_\mu\phi_1^* + \partial_\mu\phi_2\partial_\mu\phi_2^* + (4\pi g)^2\phi_1\phi_2\phi_1^*\phi_2^*\right]$$

Planar graphs all look the same



**Integrability** less mysterious here
(it follows from properties of quartic vertex)

[Zamolodchikov'80]
[Isaev'03]
[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]
[Chicherin,Kazakov,Loebbert,Muller,Zhong'16]

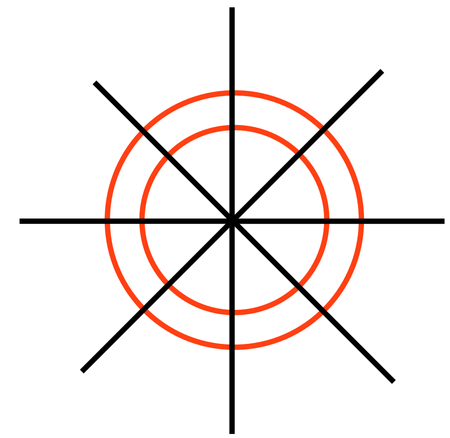
**Win** simplicity (fewer graphs, fewer particles)

**Lose** unitarity

# Fishnet factory

**Fishnet graphs** are generalized ladders

Ex 1. Periods of wheel diagrams

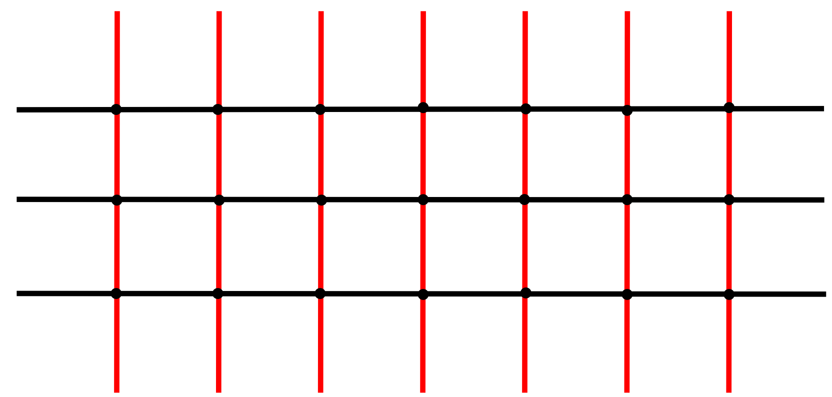


[Broadhurst'85]
[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]
[Gromov,Kazakov,Korchemsky Negro,Sizov'17]

*toy model for "Feynman integral transcendental number theory"*

[Chicherin,Kazakov, Loebbert,Muller,Zhong'17]

Ex 2. Fishnet amplitudes-correlators



[Grabner,Gromov, Kazakov,Korchemsky'17]

*toy model for various functions bootstrap program*

[Caron-Huot,Dixon,Von Hippel, McLeod,Papathanasiou'18]

*toy model for elliptic integrals (train tracks)*

[Bourjaily,He,McLeod, Von Hippel,Wilhelm'18]

Direct map between integrability picture and graphs

# A few examples

# Wheel graphs

Simplest local operator

$$\mathcal{O}_L = \mathrm{tr}\,\phi_1^L$$

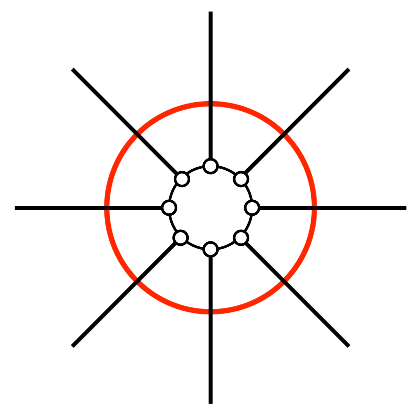
Anomalous dimension

$$\Delta = \Delta_L(g) = L + \gamma$$

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]
[Gromov,Kazakov,
Korchemsky,Negro,Sizov'17]

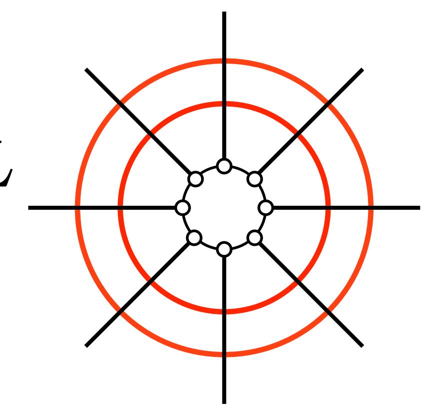
**Loop diagrams:** wheel graphs with L spokes

$$Z \;=\; 1 \;+\; g^{2L}$$



1 wheel

$$+\; g^{4L}$$



2 wheels

$$+\;\cdots$$

wave-function
renormalization

Z depends on UV cut off

$$R \sim \log \Lambda_{UV}$$

Logarithmic dependence is controlled by the anomalous dimension

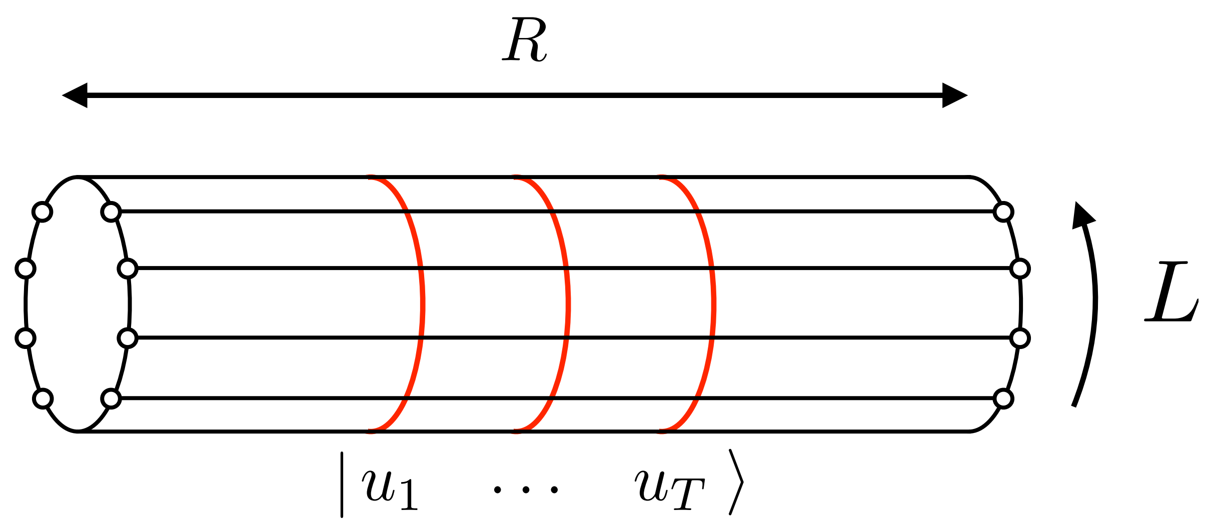
$$\log Z \sim -\gamma \times R$$

# Magnon picture

**Integrability:** particular quantum mechanical way of looking at the graphs

Grand canonical partition function on $\mathbb{R} \times S_L$

$$\mathcal{Z}_{L,R} = \sum_{T \geqslant 0} g^{2LT} \times$$




- Each wheel maps to a **magnon**

- Each **magnon** carries a momentum **u**, along horizontal direction, and a spin **a**, for harmonics on 3-sphere

- Scaling dimension = free energy

$$\log \mathcal{Z}_{L,R} = -\Delta_L(g) R + O(R^0)$$

# Thermodynamic Bethe Ansatz

Factorized scattering allows us to write TBA eqs
(where integrability helps)

[Yang,Yang'60s]
[Zamolodchikov'90s]
[Klassen-Melzer'90s]
[Ahn,Bajnok,Bombardelli,Nepomechie'11]

$$\log Y_a(u) = Lh - L\epsilon_a(u) + \sum_b \mathcal{K}_{ab} * \log\left(1 + Y_b(u)\right) + \dots$$

- length **L** of operator acts as inverse temperature
- **coupling** enters as chemical potential only
- **dynamical** input: energy of magnon & scattering kernel

$$h = \log g^2$$

$$\epsilon_a(u) = \log\left(u^2 + a^2/4\right)$$

Prediction: scaling dimension to all orders

$$\Delta = L - 2\sum_a \int \frac{du}{2\pi} \log\left(1 + Y_a(u)\right)$$

(should re-sum all the wheels)

Alternative to TBA : Baxter equations [Gromov,Kazakov,Korchemsky,Negro,Sizov'17]

# Recovering graphs

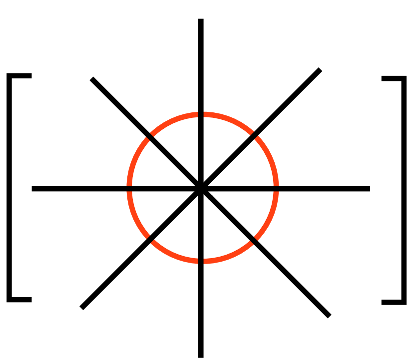
Wheel expansion (large L / weak coupling)
corresponds to low-T expansion of TBA (iterative solution)

LO $\quad Y_a(u) \simeq a^2 e^{L(h-\epsilon_a)} \ll 1$

(Boltzmann weight free magnon)

LO $\quad \Delta = L - 2\sum_a \int \frac{du}{2\pi} Y_a(u) + \dots$

Match divergent part of 1-wheel graph

div  $= -\sum_{a\geqslant 1} a^2 \int \frac{du}{\pi} \frac{g^{2L}}{(u^2 + a^2/4)^L} \propto g^{2L}\zeta(2L-3)$

[Broadhurst'85]
[Gurdogan,Kazakov'15]
[Ahn,Bajnok,Bombardelli,Nepomechie'11]

# Higher wheels

Higher loops at given length can be generated more easily using the Baxter equations

E.g. scaling dimension for L=3 up to 4 wheels (12 loops)

$$
\begin{aligned}
&\Delta_3 - 3 = -12\zeta_3\xi^6 + \xi^{12}\left(189\zeta_7 - 144{\zeta_3}^2\right) \\
&+\xi^{18}\left(-1944\zeta_{8,2,1} - 3024{\zeta_3}^3 - 3024\zeta_5{\zeta_3}^2 + 6804\zeta_7\zeta_3 + \frac{198\pi^8\zeta_3}{175} + \frac{612\pi^6\zeta_5}{35} + 270\pi^4\zeta_7 + 5994\pi^2\zeta_9 - \frac{925911\zeta_{11}}{8}\right) \\
&+\xi^{24}\left(-93312\zeta_3\zeta_{8,2,1} + \frac{10368}{5}\pi^4\zeta_{8,2,1} + 5184\pi^2\zeta_{9,3,1} + 51840\pi^2\zeta_{10,2,1} - 148716\zeta_{11,3,1} - 1061910\zeta_{12,2,1}\right. \\
&+62208\zeta_{10,2,1,1,1} - 77760{\zeta_3}^4 - 145152\zeta_5{\zeta_3}^3 - \frac{576}{7}\pi^6{\zeta_3}^3 - 864\pi^4\zeta_5{\zeta_3}^2 - 2592\pi^2\zeta_7{\zeta_3}^2 + 244944\zeta_7{\zeta_3}^2 \\
&+186588\zeta_9{\zeta_3}^2 + \frac{9504}{175}\pi^8{\zeta_3}^2 - 2592\pi^2{\zeta_5}^2\zeta_3 + \frac{29376}{35}\pi^6\zeta_5\zeta_3 + 298404\zeta_5\zeta_7\zeta_3 + 12960\pi^4\zeta_7\zeta_3 + 287712\pi^2\zeta_9\zeta_3 \\
&-5555466\zeta_{11}\zeta_3 + \frac{2910394\pi^{12}\zeta_3}{2627625} + 57672{\zeta_5}^3 - 71442{\zeta_7}^2 + \frac{13953\pi^{10}\zeta_5}{1925} + \frac{7293\pi^8\zeta_7}{175} - \frac{19959\pi^6\zeta_9}{5} \\
&\left.+\frac{119979\pi^4\zeta_{11}}{2} + \frac{10738413\pi^2\zeta_{13}}{2} - \frac{4607294013\zeta_{15}}{80}\right) + O\left(\xi^{30}\right),
\end{aligned}
$$

where $\zeta_{i_1,\ldots,i_k} = \sum_{n_1>\cdots>n_k>0} 1/(n_1^{i_1}\ldots n_k^{i_k})$ are multiple zeta functions

[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]

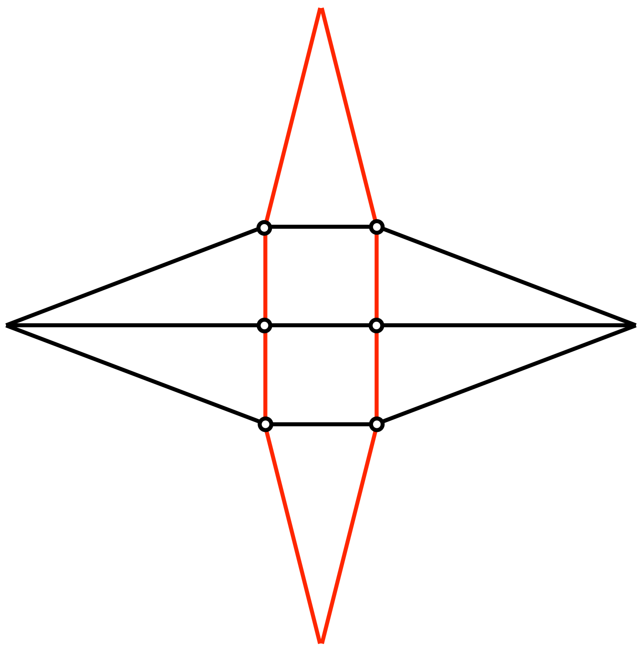
coupling $\xi^2 = g^2$

# Fishnet 4pt function

[BB,Dixon'17]

$$G_{n,m} = \langle \phi_2^n(x_1)\phi_1^m(x_3)\phi_2^{\dagger n}(x_2)\phi_1^{\dagger m}(x_4)\rangle$$

$$= \frac{g^{2mn}}{(x_{12}^2)^n(x_{34}^2)^m} \times \Phi_{m,n}(u,v)$$




**UV-IR finite**, depends on 2 cross ratios **u** and **v**

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}$$

$$v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} = \frac{1}{(1-z)(1-\bar{z})}$$

Parameterization
(for $n \geqslant m$ )

$$\Phi_{m,n}(u,v) = \left[\frac{(1-z)(1-\bar{z})}{z-\bar{z}}\right]^m \times I_{m,n}(z,\bar{z})$$

with $I_{m,n}$ a pure function (i.e. iterated integral) of weight $2mn$

# Fishnet 4pt function

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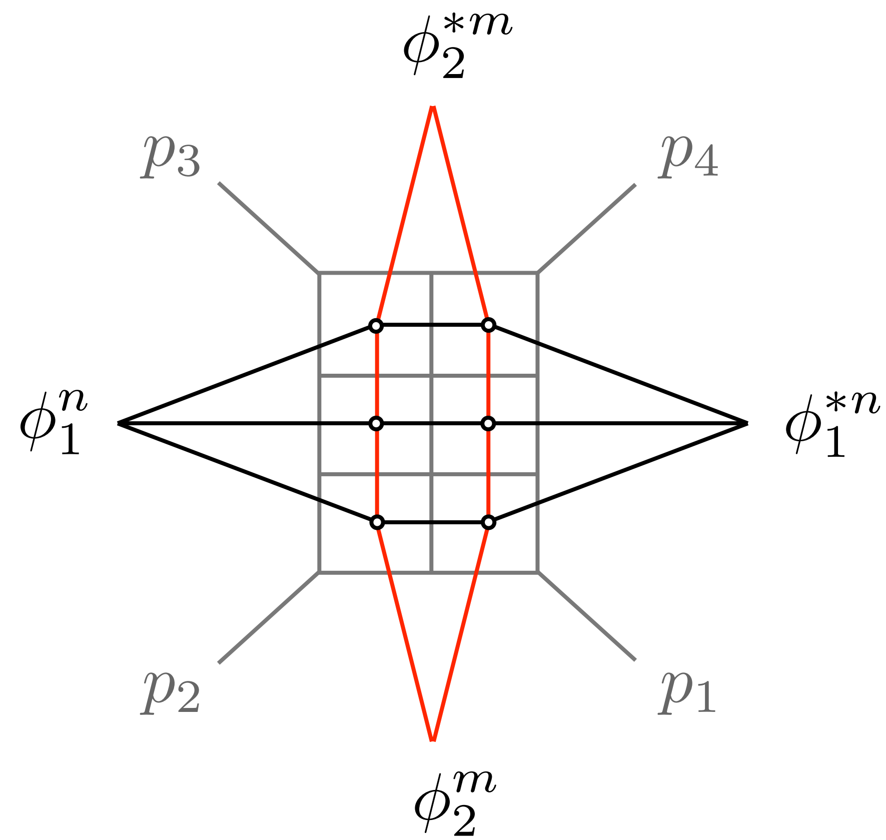
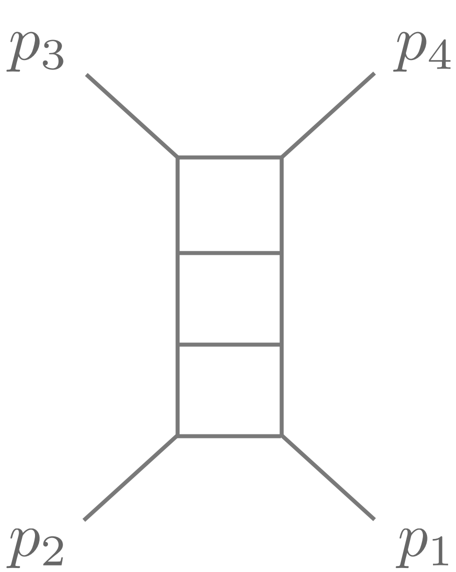



**UV-IR finite**, depends on 2 cross ratios **u** and **v**

**Ladders** when m = 1 [Usyukina,Davydychev'93]




$$I_{1,n}(z,\bar z) = L_n(z,\bar z)$$

Weight 2n generalization of **Bloch-Wigner** dilogarithm

$$L_n(z,\bar z) = \sum_{j=n}^{2n} \frac{j![-\log(z\bar z)]^{2n-j}}{n!(j-n)!(2n-j)!}[\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar z)]$$

# Fishnet 4pt function

$$G_{n,m} = \langle \phi_2^n(x_1)\phi_1^m(x_3)\phi_2^{\dagger n}(x_2)\phi_1^{\dagger m}(x_4)\rangle$$

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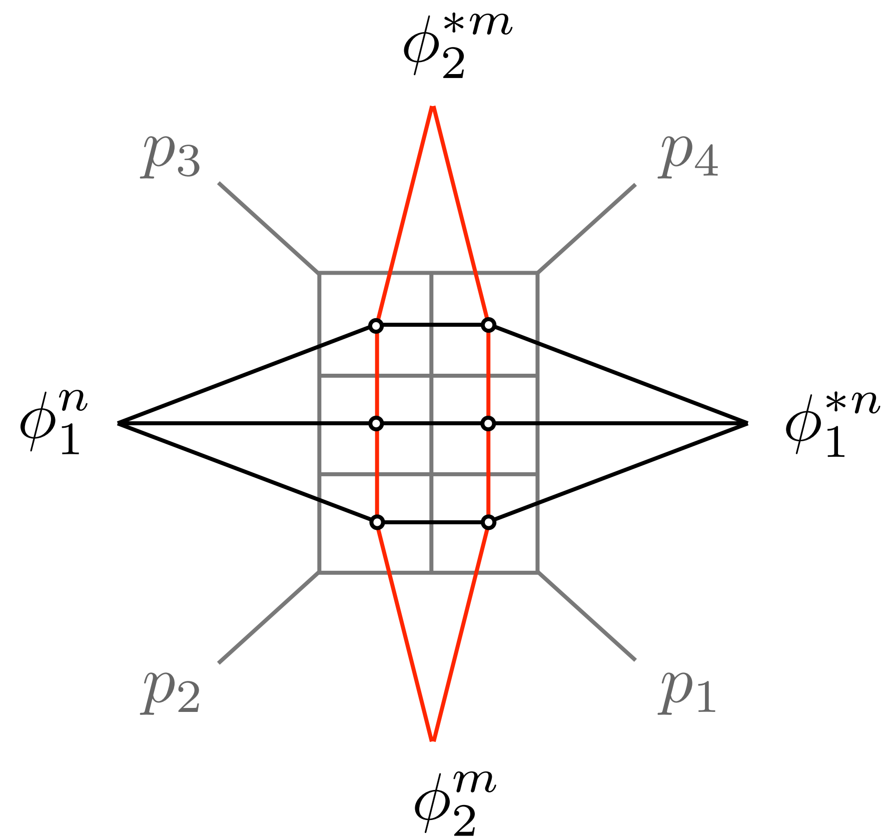



**UV-IR finite**, depends on 2 cross ratios **u** and **v**

What are these Feynman integrals equal to for higher m?

# Integrability

Cut 4pt function into two hexagons ~ triangles

[BB,Komatsu,Vieira'15]
[Fleury,Komatsu'16]
[Eden,Sfondrini'16]

Magnon picture

Vertical lines map to reference state of length n

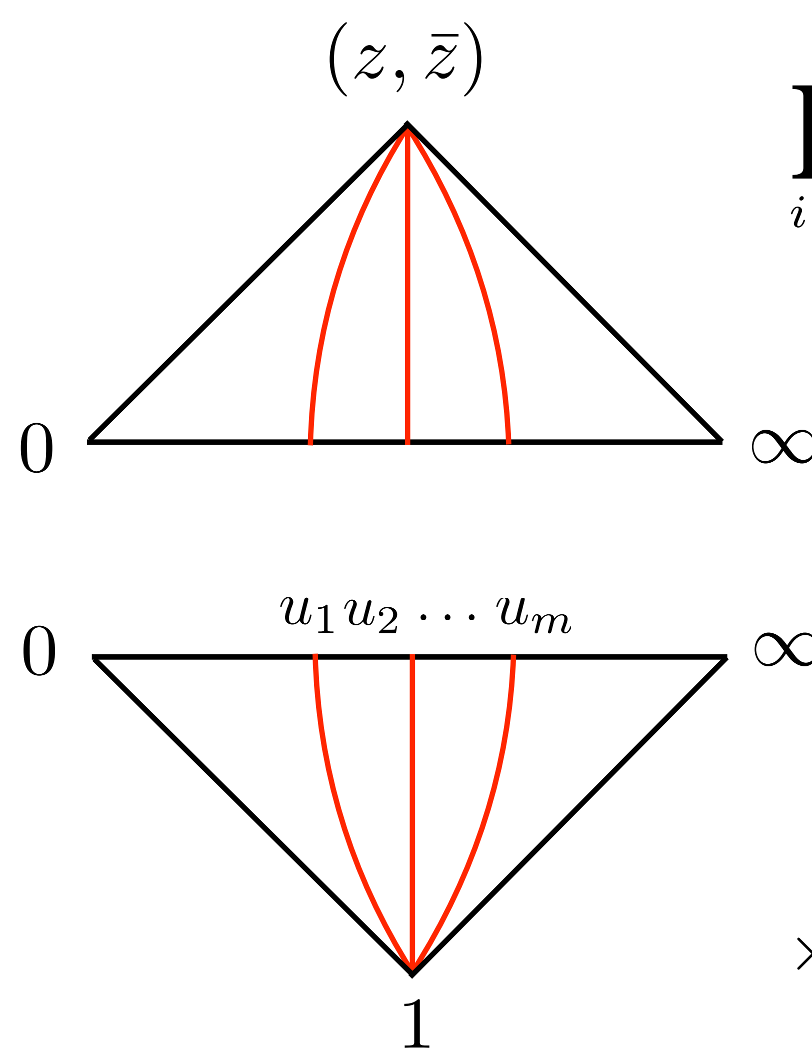
Horizontal lines map to m **magnons** propagating through it

Hexagon factorization

annihilation of m magnon state on top triangle

propagation of state from one triangle to another

creation of m magnon state on bottom triangle




$$\prod_{i>j} h_{a_i a_j}(u_i,u_j)$$

$$\times \prod_{i=1}^{m} \frac{z^{\frac{1}{2}-iu_i+\frac{a_i}{2}}\,\bar{z}^{\frac{1}{2}-iu_i-\frac{a_i}{2}}}{(z-\bar{z})}$$

(character = geometry)

$$\times \prod_{i=1}^{m} \mu_{a_i}(u_i)\, e^{-nE_{a_i}(u_i)}$$

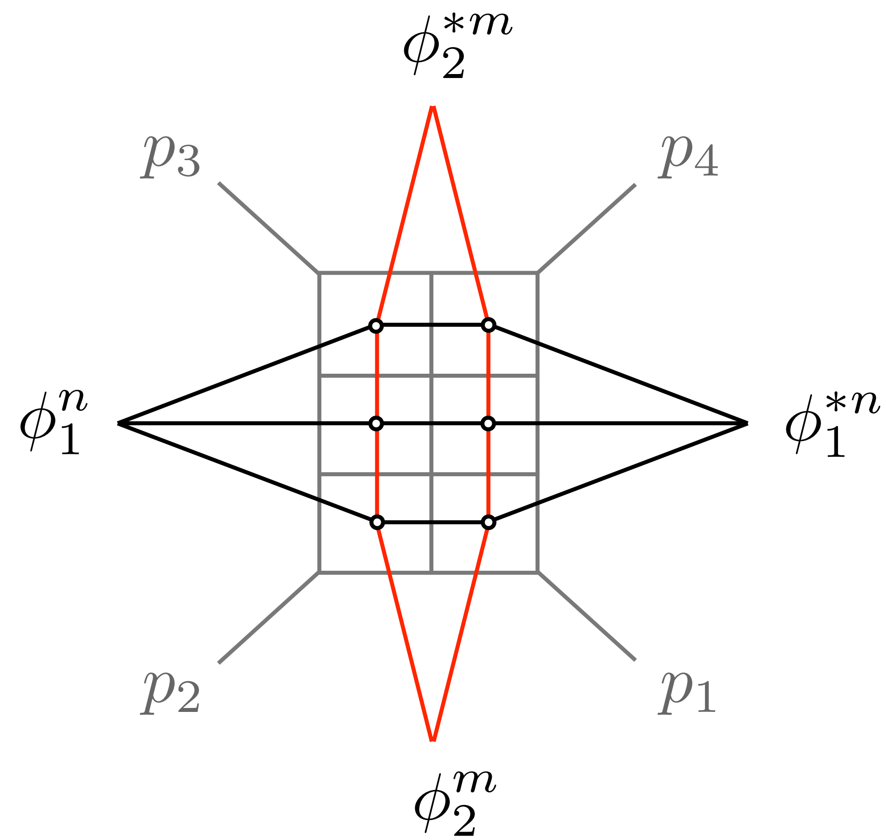
$$\times \prod_{i<j} h_{a_i a_j}(u_i,u_j)$$

# Fishnet integral as a determinant

$$G_{n,m} = \langle \phi_2^n(x_1)\phi_1^m(x_3)\phi_2^{\dagger n}(x_2)\phi_1^{\dagger m}(x_4)\rangle$$

$$= \frac{g^{2mn}}{(x_{12}^2)^n(x_{34}^2)^m} \times \Phi_{m,n}(u,v)$$

**UV-IR finite**, depends on 2 cross ratios **u** and **v**

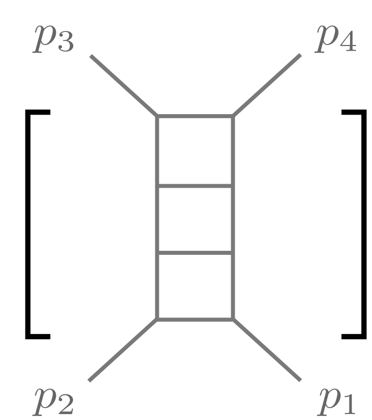



[BB,Dixon'17]

**Integrability** provides matrix-model-like integral representations

**Analyticity** requirements (Steinmann's relations)

$\longrightarrow$ weight 2mn pure function $\quad I_{m,n} \propto \quad \det$

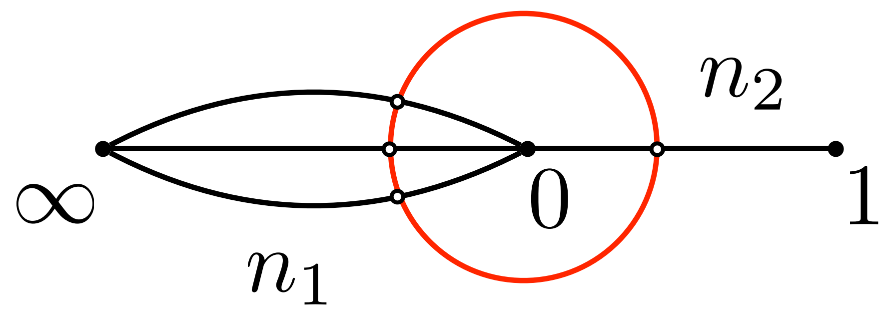



Determinant of a m-by-m Hankel matrix M of **ladders**

$$M_{ij} = (n-m+i+j-2)!(n-m+i+j-1)! \times L_{n-m+i+j-1}(z,\bar{z})$$

# More integrability

Hexagons can also be used to compute 3pt functions [BB,Caetano,Fleury'18]




Limitation on number of magnons (red lines) due to divergences

Divergences can be consistently removed for 1 magnon = 1 wheel graph

| $n_1$ | $n_2$ | $C_{\text{1-wheel}}$ |
|---|---|---|
| 1 | 2 | $6\zeta_3$ |
| 1 | 3 | $20\zeta_5$ |
| 1 | 4 | $70\zeta_7$ |
| 1 | 5 | $252\zeta_9$ |
| 2 | 2 | $-6\zeta_3^2 + 20\zeta_5$ |
| 2 | 3 | $-30\zeta_3\zeta_5 + 70\zeta_7$ |
| 2 | 4 | $-10\zeta_5^2 - 112\zeta_3\zeta_7 + 252\zeta_9$ |
| 3 | 3 | $-290\zeta_5^2 + 112\zeta_3\zeta_7 + 252\zeta_9$ |
| 3 | 4 | $-1176\zeta_5\zeta_7 + 420\zeta_3\zeta_9 + 924\zeta_{11}$ |
| 4 | 4 | $-3178\zeta_7^2 - 1680\zeta_5\zeta_9 + 1584\zeta_3\zeta_{11} + 3432\zeta_{13}$ |

# Large fishnet graphs

# Continuum limit

**Duality** with string in AdS?

Fishnet limit forces 't Hooft coupling of N=4 to be small

Tensionless string in AdS?

**Insight** from 70's: connection between fishnets and string WS

[Nielsen,Olesen'70]
[Fairlie,Nielsen'70]
[Sakita,Virasoro'70]

[Zamolodchikov'80]

Volume 97B, number 1 PHYSICS LETTERS 17 November 1980

"FISHING-NET" DIAGRAMS AS A COMPLETELY INTEGRABLE SYSTEM

A.B. ZAMOLODCHIKOV
*The Academy of Sciences of the USSR, L.D. Landau Institute for Theoretical Physics, Chernogolovka, USSR*

Received 29 July 1980

The "fishing-net" planar Feynman diagrams with massless scalar propagators are shown to be equivalent to some completely integrable lattice statistical system. The infinite-volume partition function for this system is computed exactly.

# Continuum limit

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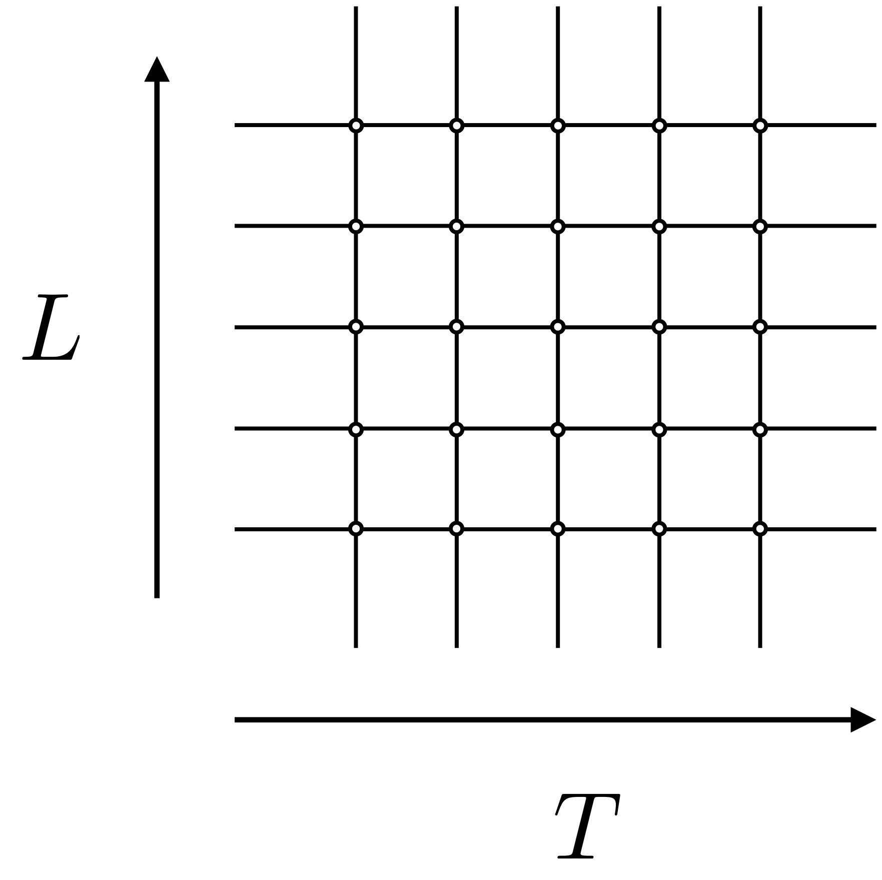
[Zamolodchikov'80]

Zamolodchikov's thermodynamic scaling

$$L, T \to \infty$$

$$\log Z_{L,T} = -L \times T \log g_{cr}^2$$

$$g_{cr} = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)} = 0.7\ldots$$



# Continuum limit

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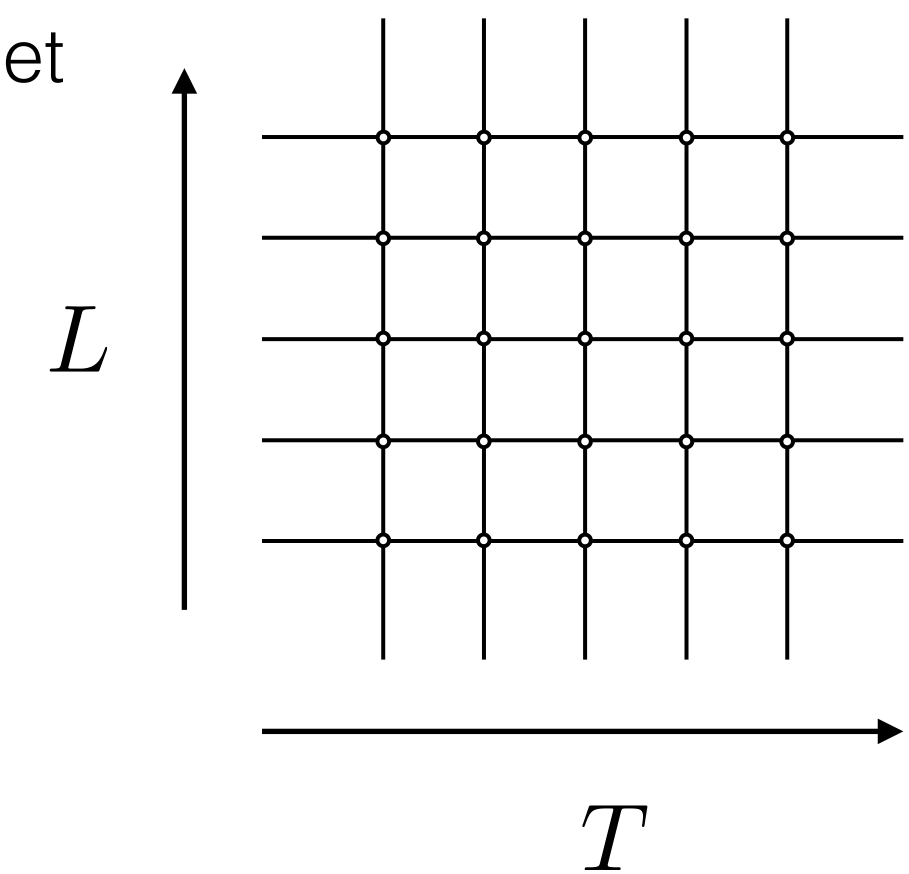
[Zamolodchikov'80]

**Meaning of $g_{cr}$ coupling in fishnet CFT?**

$L, T \to \infty$

critical value of coupling constant for which fishnet observables dominated by "dense graphs"




**World-sheet theory of fishnets?**

Claim: 2d non-linear AdS5 sigma model

(Graphs live in 4d but target space is 5d; as in conventional AdS/CFT)

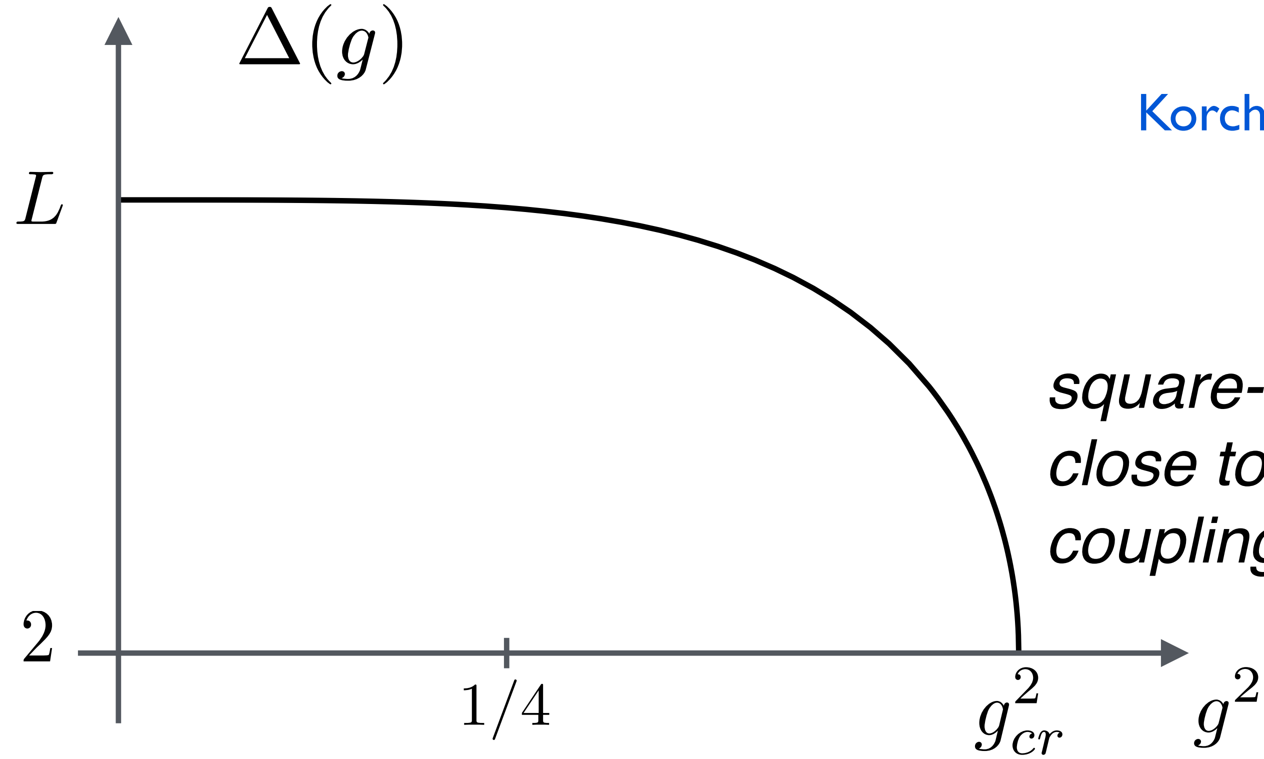
# Illustration

Look at scaling dimension $\Delta$ of BMN vacuum operator $\mathcal{O}_L = \mathrm{tr}\,\phi_1^L$

Thermodynamic limit

$$L \to \infty$$

$$\Delta \sim L f(g)$$




*square-root behaviour close to critical coupling*

see also [Gromov,Kazakov, Korchemsky,Negro,Sizov'17]

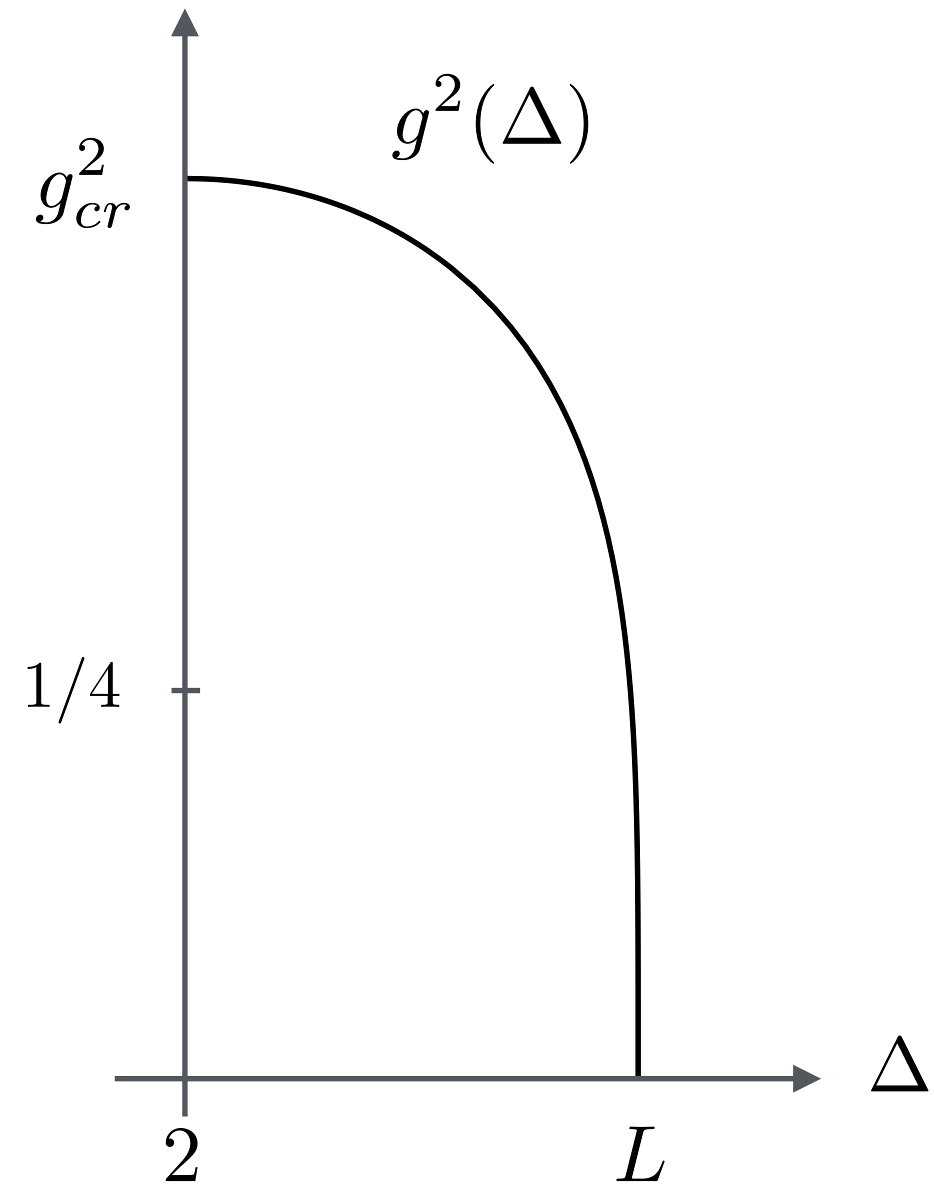
# Dual (rotated) view

Look at scaling dimension $\Delta$ of BMN vacuum operator $\mathcal{O}_L = \mathrm{tr}\,\phi_1^L$

Thermodynamic limit

$$L \to \infty$$

$$\Delta \sim L f(g)$$




Fishnet graph = AdS5 sigma model

with the dictionary

BMN operator = tachyon $\quad \mathrm{tr}\,\phi_1^L \leftrightarrow V_\Delta \sim e^{-i\Delta t}$

coupling = worldsheet energy $\quad \log g^{2L} = \log g_{cr}^{2L} + E_{2d}(\Delta, L)$

# Thermodynamic limit

Interesting when chemical potential is bigger than mass of lightest magnon

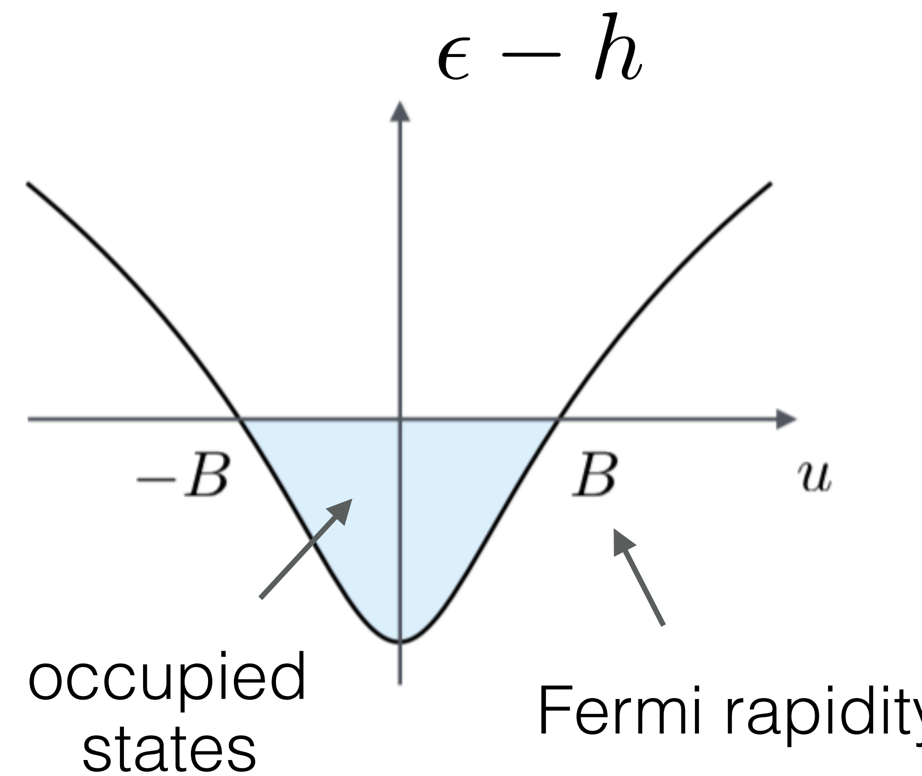
$$h = \epsilon(u = 0) = \log 1/4$$

i.e. $$g > 1/2$$

A **Fermi sea** forms

All states below the Fermi rapidity are filled

Increasing coupling amounts to increasing B




Comment: truncate to s-wave (lightest) magnons
(higher harmonics decouple)

# Fredholm equation

TBA reduce to a single linear integral equation for the energy distribution

$$\chi(u) = C - \epsilon(u) + \int_{-B}^{B} \frac{du}{2\pi} \mathcal{K}(u-v)\chi(v)$$

- BC $\chi(u = \pm B) = 0$
- Chemical potential $C = \log g^2 - \int_{-B}^{B} \frac{du}{2\pi} k(u)\chi(u)$
- Kernel $\mathcal{K}(u) = 2\psi(1+iu) + 2\psi(1-iu) + \frac{2}{1+u^2}$
- Scaling dimension $\Delta/L = 1 - \int_{-B}^{B} \frac{du}{\pi} \chi(u)$

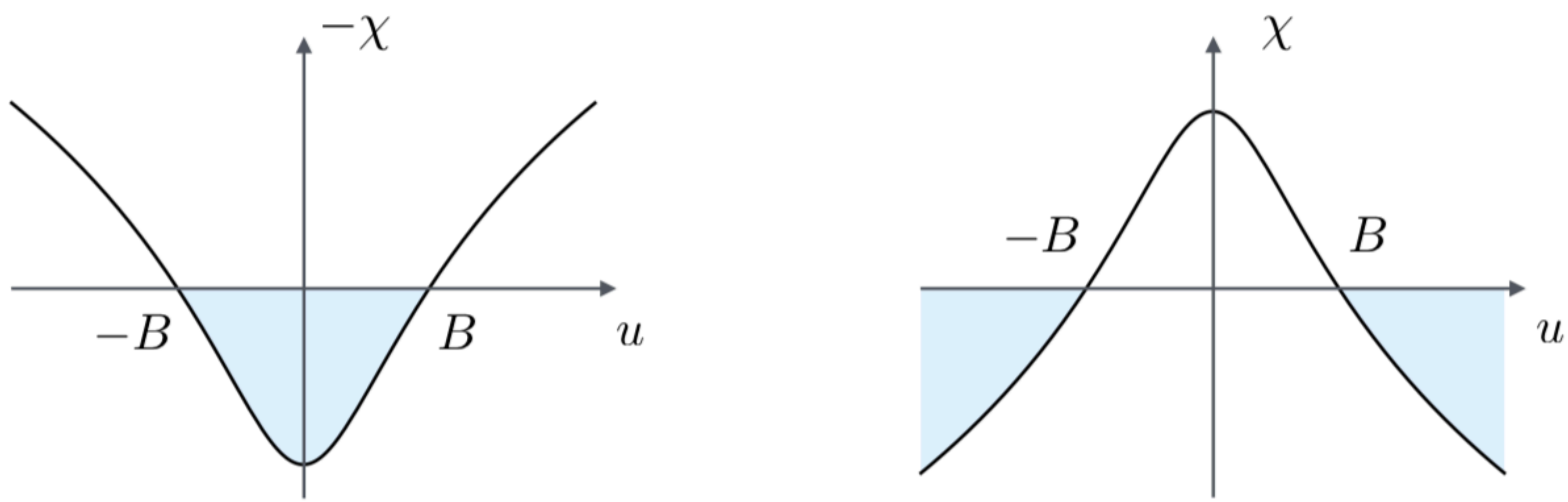
# Critical point and dualization

**Critical coupling:** large magnon density $B \to \infty$

All energy levels are filled, equation solved by Fourier transform
Zamolodchikov prediction is reproduced

$$g_{cr} = \Gamma(3/4)/\sqrt{\pi}\Gamma(5/4)$$

Particle-hole transformation




Fermi sea of magnons

dual Fermi sea

Mathematically:

1) inverting kernel $$K = -\frac{\mathcal{K}}{1 - \mathcal{K}*}$$

2) acting on both sides of the equation with $$1 - K*$$

# Dual system

Equation

$$\chi(\theta) = E(\theta) + \int_{\theta^2 > B^2} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta')$$

Energy formula

$$\log g^2 = \log g_{cr}^2 + \int_{\theta^2 > B^2} \frac{d\theta}{2\pi} P'(\theta)\chi(\theta)$$

No chemical potential
but extra BC (at $\infty$)

$$\chi(\theta) \sim -2\rho \log\theta \qquad \rho = \Delta/L =$$ charge density

1) Kernel

$$K(\theta) = \frac{\partial}{i\partial\theta} \log \frac{\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{i\theta}{2\pi})\Gamma(\frac{1}{4}-\frac{i\theta}{2\pi})}{\Gamma(1-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}-\frac{i\theta}{2\pi})\Gamma(\frac{1}{4}+\frac{i\theta}{2\pi})}$$

2) Dispersion relation

$$E(\theta) = \chi_{cr} \sim \frac{m}{2} e^{-|\theta|} \qquad P(\theta) = -iE(\theta + i\pi/2)$$

with $m = 4\sqrt{2}$ a mass scale

# Interpretation

What is the dual system describing?

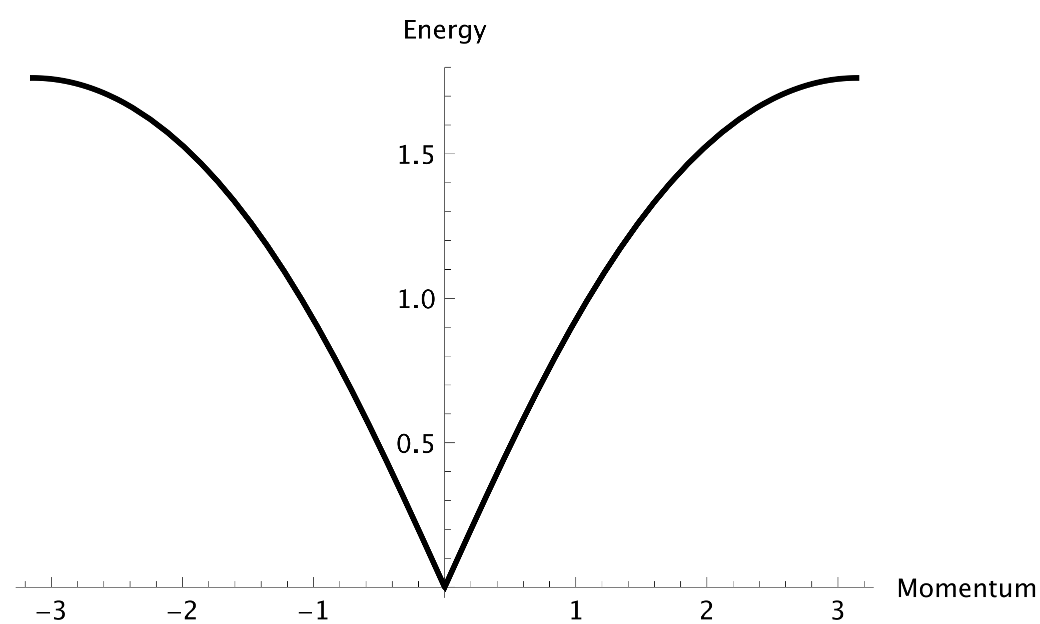
1) **Kernel**: $$K = -i\partial_\theta \log S_{O(6)}$$

Particles scatter as in 2d O(6) non-linear sigma model!

[Zamolodchikov&Zamolodchikov'78]

2) **Dispersion relation**:

$$\sinh^2(\frac{1}{2}E) = \sin^2(\frac{1}{2}P)$$




Gapless excitations (unlike O(6) model)

E decreases when $\theta$ increases
Support is non-compact & density is not normalizable (cannot count excitations)

No mass gap + continuous spectrum
Suggest sigma model with **non-compact** target space
Proposal: integrable lattice completion of $AdS_5$ sigma model

# Dual theory

**2d sigma model** on hyperbolic target space $-Y_0^2 + Y_\perp^2 - Y_{d+1}^2 = -1$

$$\mathcal{L} = -\frac{1}{2e^2} G^{AB} \partial_a Y_A \partial^a Y_B$$

Weakly coupled for large radius $R_{AdS}^2 \sim 1/e^2 \gg 1$

**Negative** curvature $\longleftrightarrow$ **Positive** beta function

coupling grows with the energy scale $\frac{1}{e^2(\mu)} = \frac{d}{2\pi} \log\left(\Lambda/\mu\right)$

1. Theory is weakly coupled in IR
2. There is no mass gap
3. Integrable but gapless and no good particle picture

Similarities with massless factorized scattering theories (although not clear what particles to scatter)

[Zamolodchikov&Zamoldchikov'92]
[Fendley,Saleur,Zamolodchikov'93]
[Fateev,Onofri,Zamolodchikov'93]

# Dual state

Take sigma model on circle of **radius L**

Interested in "ground state": **tachyon** $V_\Delta \sim e^{-i\Delta t}$

(extremum of energy E for given charge $\Delta$ )

Classical energy

$$E = -\frac{e^2\Delta^2}{2L}$$

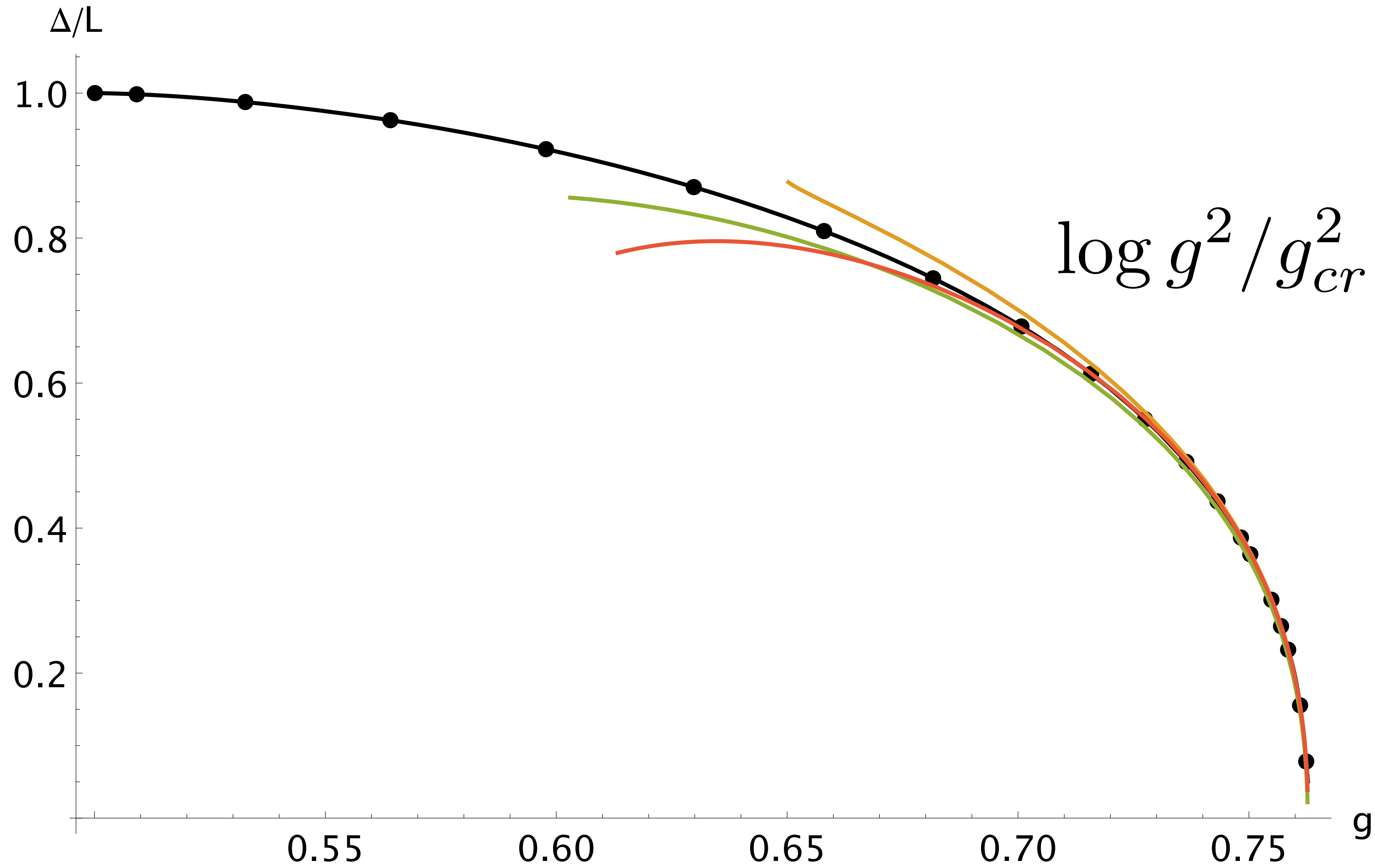
Same as in O(d+2) model if not for the sign of the coupling $e^2 \leftrightarrow -e^2$

**Quadratic scaling near critical point (** $\Delta \to 0$ **)**

(robust perturbatively; loops translate into logarithmic corrections)

# Numerics

SM prediction fits numerical sol. of linear eq. near critical point




$$\log g^2/g_{cr}^2 = -\frac{e^2\Delta^2}{2L^2}$$

WS PT
at 1st, 2nd
and 3rd
order

$$g_{\mathrm{thr}} = 0.5 \qquad\qquad g_{\mathrm{cr}} = 0.76276$$

thermodynamic window

# All order argument

Dual equation describes tachyon energy of the AdS sigma model to **all orders** in perturbation theory

Solutions for sphere and hyperboloid can be constructed to any order in $1/B$ expansion [Volin'09]
They become identical if one flips the sign of the Fermi rapidity

$$B \to -B$$

Fermi rapidity acts as inverse running coupling

$$B \sim 1/e^2 \sim \log\left(L/\Delta\right) \gg 1$$

at energy scale $\sim \rho = \Delta/L$

Changing sign of B has the same effect as changing the sign of radius of curvature of the space, mapping sphere to hyperboloid

# Dual descriptions

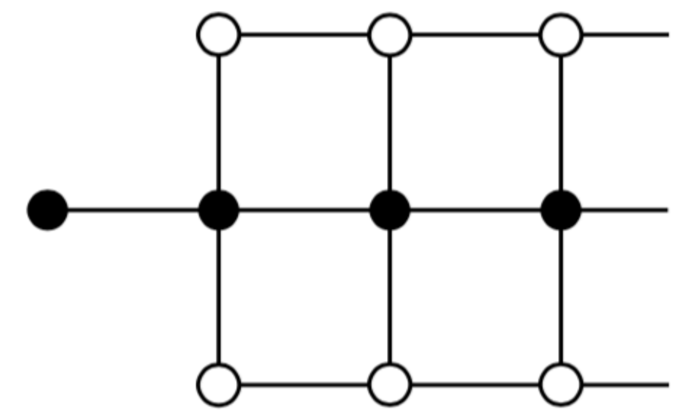
## Original magnon TBA (massive)

$$\log Y_1 = L \log g^2 - L\epsilon + \mathcal{K} * \log\left(1 + Y_1\right) + \ldots$$

$$\Delta = L - \sum_{a=1}^{\infty} \int \frac{du}{\pi} \log\left(1 + Y_a\right)$$

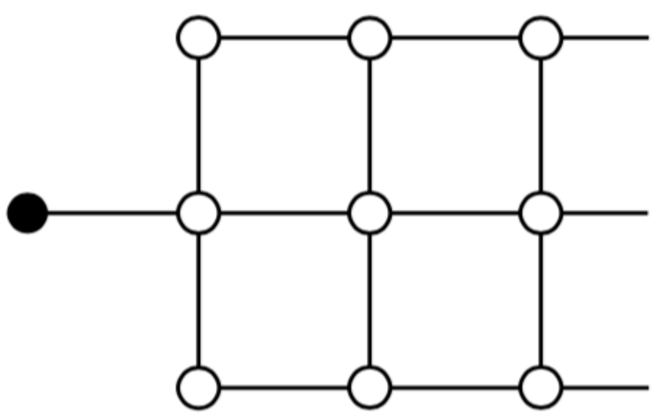
**input**: coupling $\log g^2$

**output**: scaling dimension $\Delta$



massive TBA

(black nodes = energy carriers)



massless TBA

(only 1 momentum carrier)

## Dual TBA (massless)

$$\log Y_1 = LE - K_{O(6)} * \log\left(1 + 1/Y_1\right) + \ldots$$

$$\log g^{2L}/g_{cr}^{2L} = -\int \frac{d\theta}{2\pi} P'(\theta) \log\left(1 + 1/Y_1\right)$$

$\Delta$ = **input** (label tachyon rep)

$\log g^2$ = **output** (sigma model energy)

# Y system

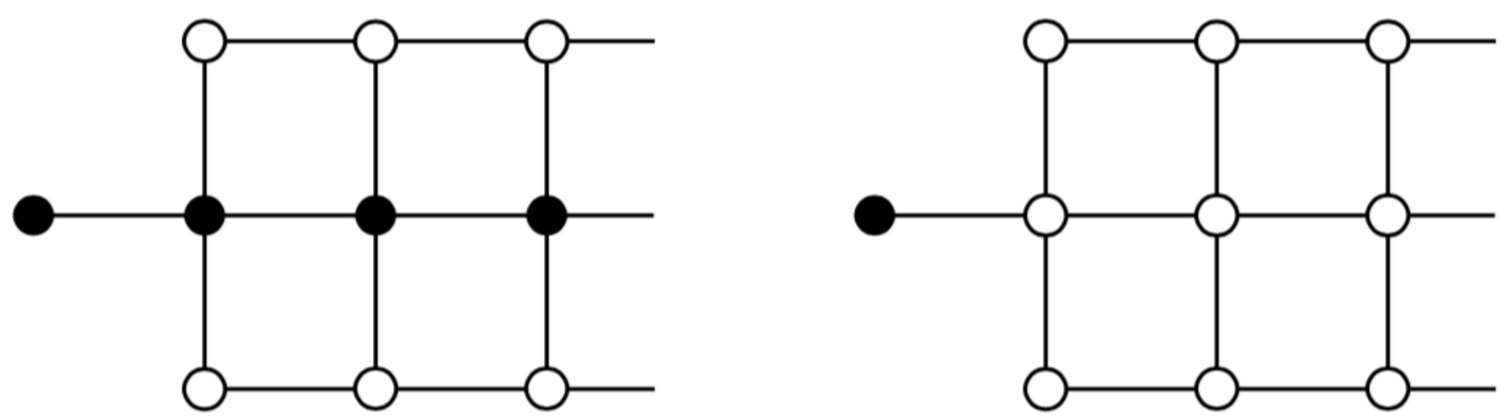
Y system equations are the same as for compact sigma model

[Balog,Hegedus'04]
[Fendley'99]

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}}=\frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

$$\frac{1}{Y_1^{++}Y_1^{--}}=(1+\frac{1}{Y_{2,1}})(1+\frac{1}{Y_{2,0}^{+}})(1+\frac{1}{Y_{2,0}^{-}})(1+\frac{1}{Y_{2,-1}})$$

regardless of the phase we use



# Tachyon energy from TBA

**TBA** analysis in CFT limit (1 / L energy or Casimir energy)

[Zamolodchikov’90s]
[Klassen-Melzer’90s]

$$L \gg 1 \qquad \Delta = O(1)$$

1) Get scale-invariant “kink” solution

2) Run dilog routine for central charge

$$c_\star = \sum_i \mathcal{L}\left(\frac{Y_i^\star}{1+Y_i^\star}\right)$$

with Rogers dilogarithm $\mathcal{L}(x) = \frac{6}{\pi^2}\left(\mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)\right)$

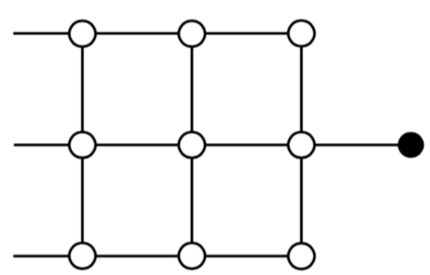
**Here, kink interpolates between:**

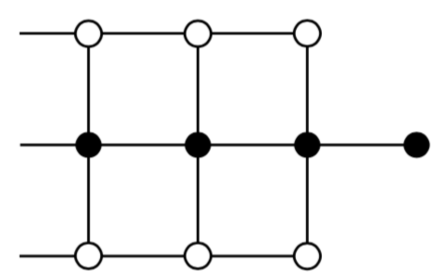
see also
[Balog,Hegedus’04]

symmetric phase

O(6)

$$c_0 = 7$$





broken phase

O(4)

$$c_\infty = 2$$

Yield **TBA central charge** $c = c_0 - c_\infty = 5$

# Tachyon energy from CFT

**CFT analysis**: close to IR fixed point (large length) for $AdS_{d+1}$

Energy level maps to anomalous dimension of vertex operator (operator-state correspondence)

$$V_\Delta \sim e^{-i\Delta t} \quad \rightarrow \quad E_{2d} = -\frac{\pi c_{eff}(L)}{6L} - \frac{e^2 \Delta(\Delta - d)}{2L} + O(e^4)$$

running coupling (distance L) $\quad e^2 \sim \dfrac{2\pi}{d \log L} \ll 1$

effective central charge (distance L): $\quad c_{eff}(L) = d + 1 + O(e^2)$

Agreement with **TBA analysis** (for d=4)

# Spinning the wheels

Operators with spin $\operatorname{tr} \partial^M \phi_1^L$

Scaling dimension at weak coupling $\Delta = L + M + O(g^{2L})$

Primaries map to solutions of Bethe equations for non-compact spin chain

$$1 = \left(\frac{v_k - i/2}{v_k + i/2}\right)^L \prod_{j \neq k}^{M} \frac{v_k - v_j - i}{v_k - v_j + i} \times e^{i\Phi_k}$$

dressing factor = long-range corrections from wheels

Anomalous dimension obtained as before

(Y's solving TBA eqs with extra source terms from the v's)

$$\gamma = -\sum_{a \geqslant 1} \int \frac{du}{\pi} \log\left(1 + Y_a(u)\right)$$

Dualize…

# Spinning the wheels

Dual energy

$$L \log g^2/g_{cr}^2 = \sum_{i=1}^{M} E(\theta_i) - \int \frac{d\theta}{2\pi}\, P'(\theta) \log\left(1 + 1/Y_1(\theta)\right)$$

transverse excitations (positive) $+$ "vacuum" energy (negative)

in agreement with expectation for states dual to

$$V_{\Delta,M,N} \sim \partial^N (Y_1 + iY_2)^M \times e^{-i\Delta t}$$

and signature of AdS

Dual Bethe equations $$e^{iP(\theta_k)L} \prod_{j\neq k}^{M} S_{O(6)}(\theta_k - \theta_j) = 1$$

as for O(6) model if not for the momentum $$P(\theta) = \mp \frac{m}{2} e^{-|\theta|}$$

if not for the momentum that decays at large rapidity

$\longrightarrow$ flip the sign of the coupling

# Deformations

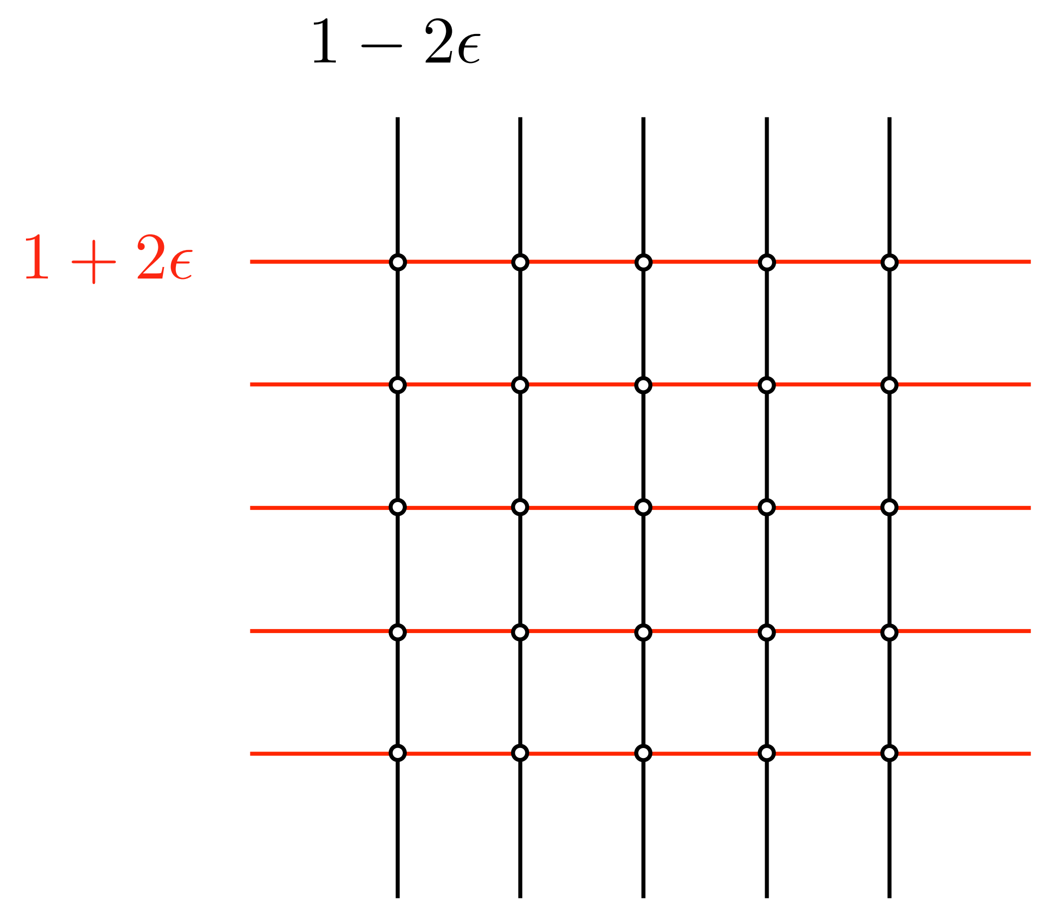
# Deformations

**Anisotropic** 4d fishnets (nonlocal field theory)

[Zamolodchikov'80]
[Kazakov,Olivucci'18]

Generalized free fields

$$\frac{1}{x^{2(1\pm 2\epsilon)}}$$




with quartic coupling

Still **conformal** and **integrable**

Does it descend from a (non-local) deformation of N=4 SYM?

# Zamolodchikov's coupling

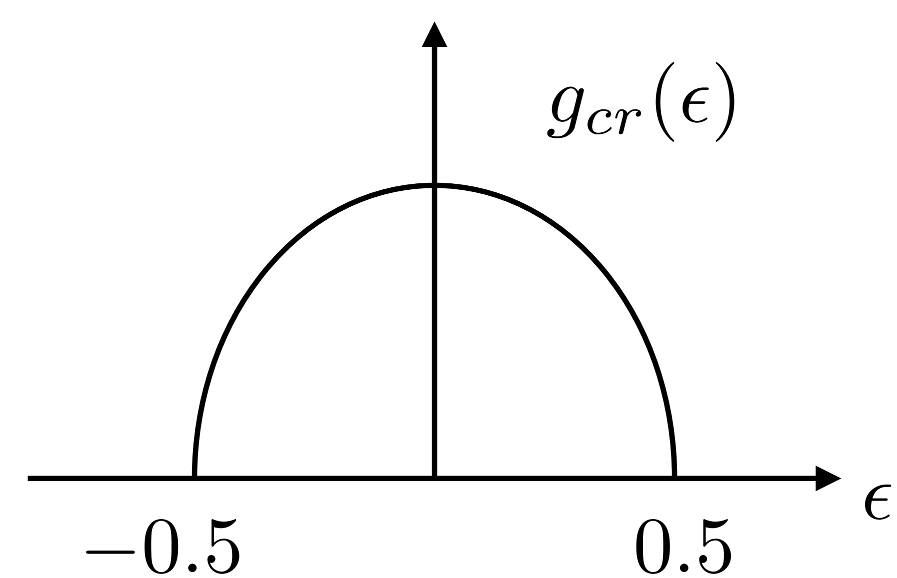
Critical coupling follows from functional equations (d=4)

- $g_{cr}^2(\epsilon) = g_{cr}^2(-\epsilon)$
- $g_{cr}^2(\epsilon) g_{cr}^2(1+\epsilon) = 16\,\epsilon(\epsilon+\frac{1}{2})^2(\epsilon+1)$

(akin to unitarity and crossing relations for S-matrix)

Minimal solution reads

$$g_{cr}^2(\epsilon) = \frac{8\epsilon\cot(\frac{\pi\epsilon}{2})\Gamma(\frac{3}{4}+\frac{\epsilon}{2})\Gamma(\frac{3}{4}-\frac{\epsilon}{2})}{\Gamma(\frac{1}{4}+\frac{\epsilon}{2})\Gamma(\frac{1}{4}-\frac{\epsilon}{2})}$$



# TBA scattering data

Where to get TBA scattering data?

Separation of variables (not easy for higher d) [Derkachov,Kazakov,Olivucci'18]

Bottom-up approach: Guess/extract building blocks
and check against field theory

E.g., 1-wheel graph

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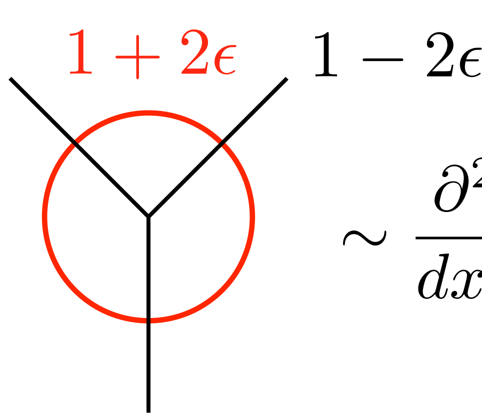


$$\sim \frac{\partial^2}{dx^2}\left( {}_3F_2[2, 1-x+2\epsilon, 1-x+2\epsilon; 2-x, 2-x; 1] \times \ldots \right)$$

# TBA scattering data

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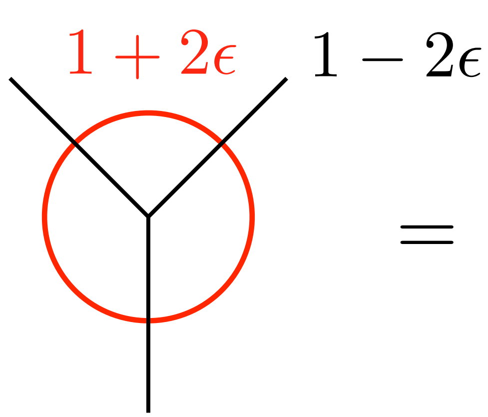
Div of 


$$= 6\zeta(3) + 180\zeta(5)\epsilon^2 + \ldots$$

Agreement with known results see e.g. [Grozin'12]

# Dual model

Reproduce Z critical coupling

$$\log g_{cr}^2 = \int \frac{du}{2\pi} k(u) \chi_{cr}(u) = \log g_Z^2$$

The rest stays the same, if not for dispersion relation

$$\sinh^2(\frac{E}{2}) = c(\epsilon)^2 \sin^2(\frac{P}{2})$$

with deformation in the speed of light $c = \tan\frac{\pi}{4}(1-2\epsilon)$

Innocuous at low energy/momentum: AdS5 sigma model, again

# Fishnet zoo

Integrable fishnets exist in any d

[Zamolodchikov'80]
[Kazakov,Olivucci'18]

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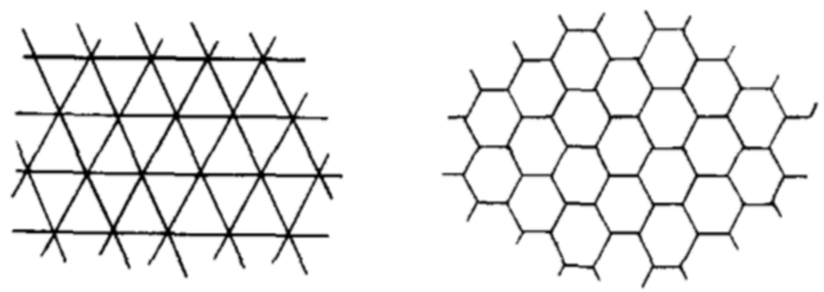

(a) (b)

Fig. 6.

shown in fig. 6a (at $\mathcal{D} = 3$) and in fig. 6b (at $\mathcal{D} = 6$)

means that the diagram with a large number $P$ of vertices behaves as $(gc)^P$ (we mean the main factor of the asymptotics), the constant $c$ being independent of the external momenta. The value of $c$ can be obtained from eq. (18) by substituting the appropriate values of $\alpha$ and $\mathcal{D}$. Thus, one obtains

$$c_4 = \left[\frac{1}{16\sqrt{\pi}}\frac{\Gamma(1/4)}{\Gamma(3/4)}\right]^2, \quad c_3 = \left[\frac{1}{12\sqrt{\pi}}\frac{\Gamma(1/6)}{\Gamma(2/3)}\right]^3,$$

$$c_6 = \left[\frac{1}{24\sqrt{\pi}}\frac{\Gamma(1/3)}{\Gamma(5/6)}\right]^{3/2}, \qquad (19)$$

Enlarged family of fishnets with Yukawa interactions, etc.

[Chicherin,Kazakov,Loebbert,Muller,Zhong'17]

General dictionary between fishnets and non-compact sigma models?

# Summary

Fishnet graphs = quantum integrable system

Fishnet graphs = 2d AdS5 sigma model

Dual model is weakly coupled when fishnet lengths are large

Large L and “small” quantum numbers = low worldsheet energy

Well tested for simplest state, but how general is the dictionary?

# String or not?

Marginality condition of sort $\quad 0 = L\mu + E_{2d}(L)$

with cosmological constant $\quad \mu = \log g_{cr}^2/g^2$

On-shell condition comes from the geometric sum over the wheels

$$\sum_{T \geqslant 0} (g/g_{cr})^{2LT} e^{-TE_{2d}(L,\Delta)} = \frac{1}{1-(g/g_{cr})^{2L} e^{-E_{2d}(L,\Delta)}}$$

with T acting as a discrete time

Is the conformal fishnet a non-critical AdS string model?

SM in UV? Follow the RG flow: small AdS in UV?

Lagrangian at lattice scale? [Gromov,Sever'19]

THANK YOU!