Scattering Amplitudes and Results in General Relativity

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Outline

- LIGO/Virgo's detection of gravitational waves has opened up new exciting possibilities for testing general relativity

  Test of general relativity in regime of strong gravity probed by merging black holes

  Urgent need for computational methods from which physical parameters can be extracted.

  Marriage of *on-shell* amplitude technology and general relativity computations seems ideal!

Outline

Viewpoint: General Relativity as a perturbative effective (quantum) field theory

New on-shell toolbox for gravity computations

New techniques for computation of physical observables in general relativity

Traditional quantization of gravity

- Known since the 1960ties that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian (Feynman, DeWitt)

- Expand Einstein-Hilbert Lagrangian :

$$\mathcal{L}_{\mathrm{E}H} = \int d^4x \left[\sqrt{-g} R \right] \qquad g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

- Derive vertices as in a particle theory - compute amplitudes as Feynman diagrams!

Quantum theory for gravity

- Gravity as a QFT-theory with self-interactions
- Non-renormalisable theory! ('t Hooft and Veltman)

Dimensionful coupling: $G_N = 1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove all UV-divergences

String theory can by introducing new length scales

Quantum gravity as an effective field theory

- (Weinberg) proposed to generalize the quantization of general relativity from the viewpoint of effective field theory

$$\mathcal{L} = \sqrt{-g}\left[\frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}}\right]$$

$$\Downarrow$$

$$\mathcal{L} = \sqrt{-g}\left\{\frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \dots\right\}$$

Effective field theory for gravity

- Consistent quantization:

  Today's viewpoint: a working perturbative low energy version of quantum gravity **(No contradiction with 'string-theory'.)**

- Applications:

  General relativity: classical limit of the EFT perturbative expansion!

  Quantum gravity: unique low energy signatures

Einstein's theory as an EFT

- Suggest general relativity augmented by higher derivative operators – the most general modified theory

  - Similar to the Standard Model – also expectation of higher derivative corrections.
  - Tiny consequences for most observables – curvature is really small.
  - Interesting to probe connections between observed bounds and theoretical predictions

Loop results for gravity

- The one-loop four point amplitude can be deduced to take the form

$$\mathcal{M} \sim \left(A + Bq^2 + \ldots + \alpha\kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots\right)$$

Focus on deriving these ~>

Short range behaviour ~ polynomials

Non-analytic long-range behavior (one-loop ~ no higher derivative contributions)

(Donoghue; NEJB, Donoghue, Holstein)

One-loop result for gravity

- The result for the amplitude (in coordinate space) after summing all diagrams is (leading in small momentum transfer contribution): (NEJB, Donoghue, Holstein)

$$-\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}+\frac{41}{10\pi}\frac{G\hbar}{r^2}\right]$$

Post-Newtonian term

New quantum term

Post-Newtonian term in complete accordance with general relativity after Born subtraction: (Iwasaki, Holstein and Ross, Neill and Rothstein, NEJB, Damgaard, Festuccia, Plante, Vanhove)

Effective field theory for gravity

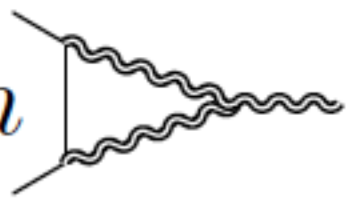
Consequence: Classical theory from loop diagrams!

Explanation: propagators involving masses in loop diagrams features cancellations of hbar factors.

(see e.g. Donoghue, Holstein)

Classical contributions from perturbative computations

- In classical gravity the long-distance terms that are related to the post-Newtonian effects are triangle diagrams (at one-loop)

m  $\sim \; m/\sqrt{-q^2}$

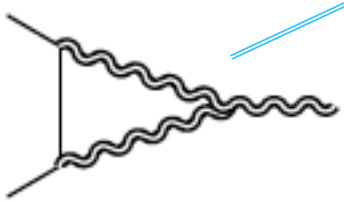
- Such contributions have cancellations of $\hbar$ and lead to purely classical terms

General relativity from loops

New derivation

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{(\ell+p_1)^2 - m_1^2 + i\epsilon}$$

$$(\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \simeq 2m_1 \ell_0$$



$$\frac{1}{2m_1} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{\ell_0 + i\epsilon}$$

General relativity from loops

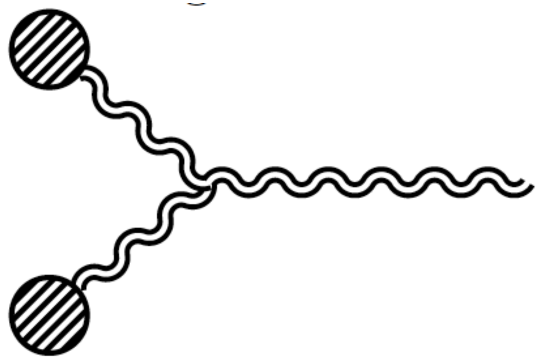
$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\frac{1}{(\ell+q)^2+i\epsilon}\frac{1}{\ell_0+i\epsilon}$$

Close contour

$$\int_{|\vec{\ell}|\ll m}\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{i}{4m}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2} = -\frac{i}{32m|\vec{q}|}$$

Interpretation

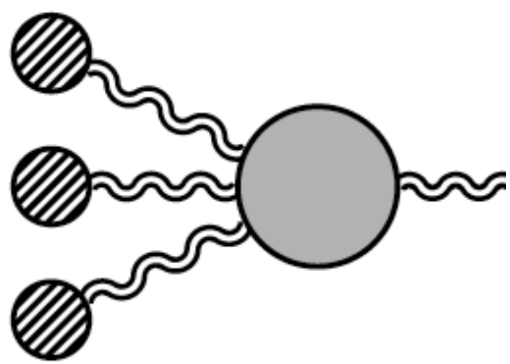
$$\int \frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2} \longleftrightarrow$$



Integration of classical sources on tree graphs – no loops!

Picture extends to higher loops

$$I_{\triangleright\triangleright(1)}(p_1,q), I_{\triangleright\triangleright(2)}(p_1,q) \leftrightarrow$$



Explains the metric computation by (Duff)

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Off-shell gravity amplitudes

- Vertices: 3pt, 4pt, 5pt,..n-pt
- Complicated expressions
- Expand Lagrangian, tedious process....

45 terms + sym

$$
\begin{aligned}
V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\
& + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\
& - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\
& + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],
\end{aligned}
$$

(DeWitt;Sannan)

Computation of perturbative amplitudes

\# Feynman diagrams: Sum over topological different diagrams

Factorial Growth!

Generic Feynman amplitude

Complex expressions involving e.g.

$(p_i \cdot p_j)$

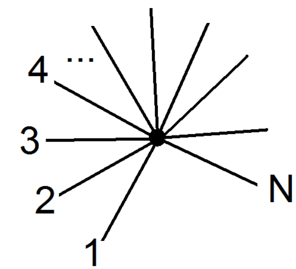
$(p_i \cdot \varepsilon_j)(\varepsilon_l \cdot \varepsilon_j)$

(no manifest symmetry or simplifications)

Amplitudes

Specifying external polarisation tensors $(\varepsilon_i \cdot \varepsilon_j)$

Colour ordering



$Tr(T_1\ T_2\ ..\ T_n)$

Simplifications

Recursion

Spinor-helicity formalism

Loop amplitudes: (Unitarity, Supersymmetric decomposition)

Inspiration from String theory

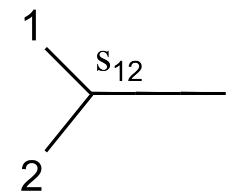
String theory

String theory given us lots of ideas..

Fact: Using (weak) string theory as a way to learn more about field theory is extremely useful..

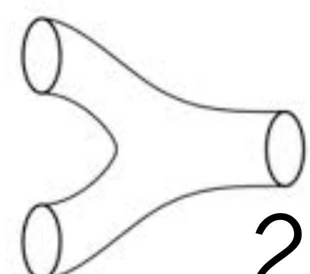
Theory

Feynman diagrams



Amplitudes

String theory



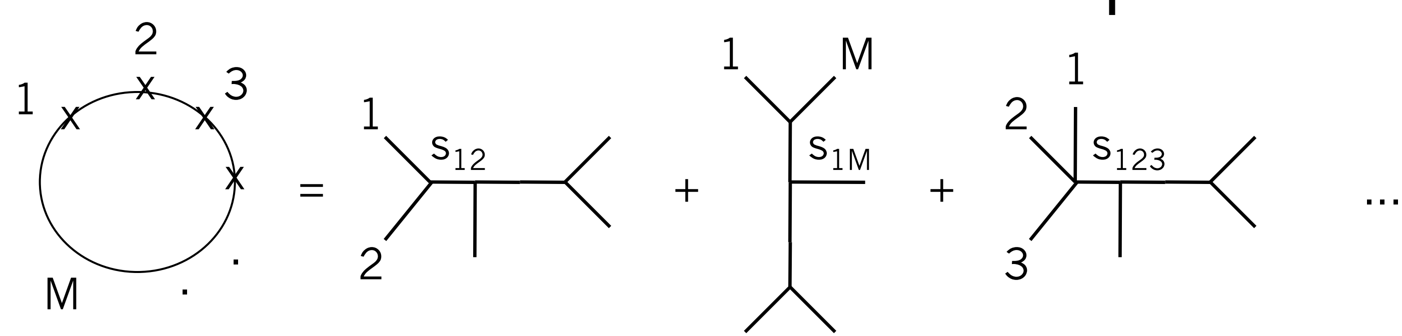
Key: String theory inspiration

Different form for amplitude

String theory adds channels up..

<->

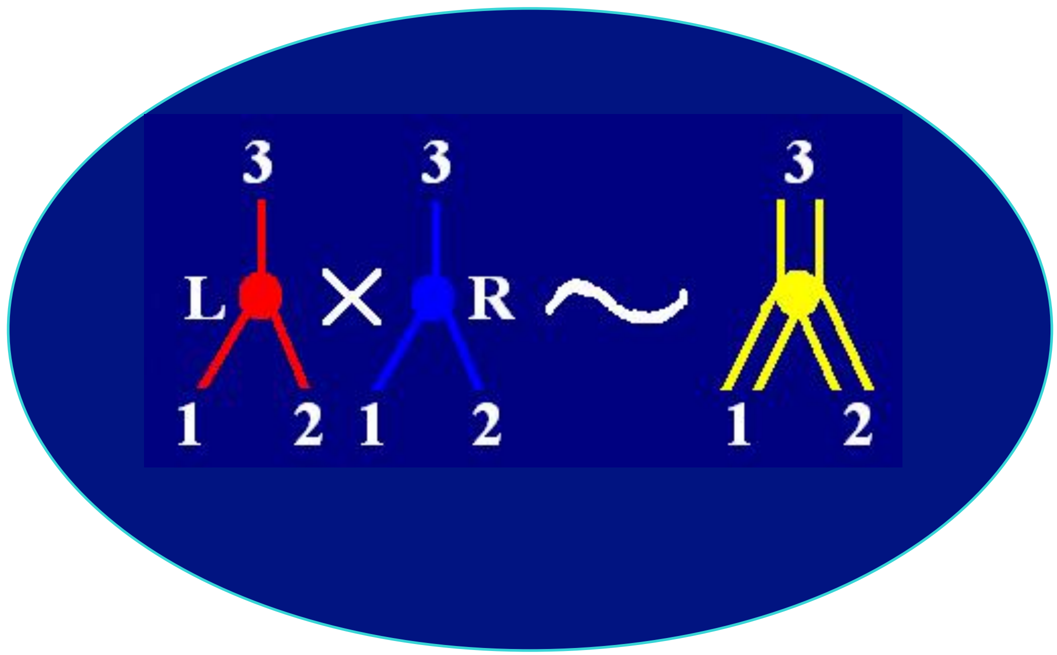
Feynman diagrams sums separate kinematic poles



Squaring relation for gravity

Gravity from (Yang-Mills)^2 (Kawai, Lewellen, Tye)

Natural from the decomposition of closed strings into open.



Gives a smart way to recycle Yang-Mills results into gravity results..
(Bern, Dixon, Perelstein, Rozowsky)

Key: on-shell states formalism

Spinor products :

$$\langle i\ j\rangle = \epsilon^{mn}\lambda^i_m\lambda^j_n \quad [i\ j] = \epsilon^{\dot m\dot n}\tilde\lambda^i_{\dot m}\tilde\lambda^j_{\dot n}$$

Different representations of the Lorentz group

$$p_{a\dot a} = \sigma^\mu_{a\dot a}p_\mu$$

$$p^\mu p_\mu = 0 \qquad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a}$$

Momentum parts of amplitudes:

$$q_{a\dot a}=\mu_a\tilde\mu_{\dot a} \quad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a} \quad 2(p\cdot q) = s_{ij} = -\langle\lambda,\mu\rangle[\tilde\lambda,\tilde\mu]$$

Spin-2 polarisation tensors in terms of helicities, (squares of those of YM):

(Xu, Zhang, Chang)

$$\varepsilon^-_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda,\tilde\mu]} \qquad \tilde\varepsilon^+_{a\dot a} = \frac{\mu_a\tilde\lambda_{\dot a}}{\langle\mu,\lambda\rangle} \qquad \begin{matrix}\varepsilon^- & \varepsilon^- \\ \tilde\varepsilon^+ & \tilde\varepsilon^+\end{matrix}$$

Miracles of MHV-amplitudes

(n) same helicities vanishes

$$A^{\text{tree}}(1^+,2^+,3^+,4^+,..) = 0$$

(n-1) same helicities vanishes

$$A^{\text{tree}}(1^+,2^+,..,j^-,..) = 0$$

(n-2) same helicities:

$$A^{\text{tree}}(1^+,2^+,..,j^-,..,k^-,..)$$

$A^{\text{tree MHV}}$ Given by the formula (Parke and Taylor) and proven by (Berends and Giele)

First non-trivial example,
(M)aximally
(H)elicity (V)iolating
(MHV) amplitudes

One single term!!

$$\boxed{i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}}$$

Simplifications from Spinor-Helicity

$$s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

Huge simplifications

$$V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa \operatorname{sym}\Big[-\frac{1}{2} P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],$$

45 terms + sym

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^{-}_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^{+}_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

Gravity: $A_3(1^-, 2^-, 3^+)$

$$\varepsilon^{-}\ \varepsilon^{-}$$
$$\tilde{\varepsilon}^{+}\ \tilde{\varepsilon}^{+}$$

$$A_3(1^-,2^-,3^+) = -i\frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

Gravity MHV amplitudes

Can be generated from KLT via YM MHV amplitudes.

$$M_4^{\text{tree}}(1^-,2^-,3^+,4^+) = i\,\langle 1\,2\rangle^8\,\frac{[1\,2]}{\langle 3\,4\rangle\;N(4)}$$

$$M_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) = i\,\langle 1\,2\rangle^8\,\frac{\varepsilon(1,2,3,4)}{N(5)}$$

Anti holomorphic
Contributions
– feature in gravity

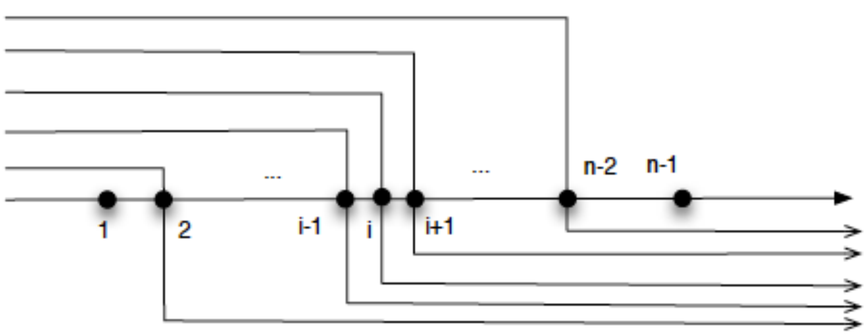
Compact notation through momentum kernel and monodromy relations

(NEJBB, Damgaard, Vanhove;
Steiberger;
NEJBB, Damgaard,
Feng, Sondergaard;
NEJBB, Damgaard,
Sondergaard,Vanhove)

KLT relations

Redoing KLT using S kernels leads to...

$$\mathcal{M}_n = (-i/4)^{n-3} \times$$
$$\sum_{\sigma}\sum_{\gamma,\beta} \mathcal{S}_{\alpha'}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2,\ldots,j-1)]_{k_1} \mathcal{S}_{\alpha'}[\beta(\sigma(j),\ldots,\sigma(n-2))|\sigma(j,\ldots,n-2)]_{k_{n-1}}$$
$$\times\, \mathcal{A}_n(1,\sigma(2,\ldots,n-2),n-1,n)\, \widetilde{\mathcal{A}}_n(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,n-1,\beta(\sigma(j),\ldots,\sigma(n-2)),n)\,.$$



Beautifully symmetric form for (j=n-1)

gravity...

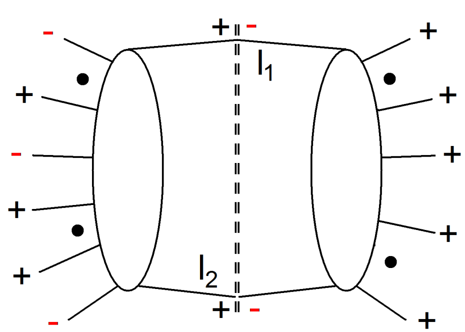
$$M_n = (-1)^n \sum_{\gamma,\beta} \frac{\widetilde{A}_n(n,\gamma_{2,n-1},1)\mathcal{S}[\gamma_{2,n-1}|\beta_{2,n-1}]_{p_1} A_n(1,\beta_{2,n-1},n)}{s_{12\ldots(n-1)}}$$

Unitarity cuts

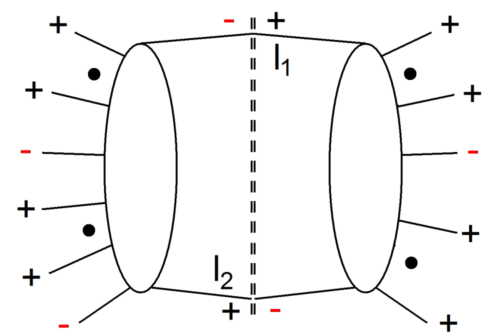
Helicity formalism require unitarity methods

$$C_{i,\dots,j} \;=\; \mathrm{Im}_{K_{i,\dots,j}>0}\, M^{\text{1-loop}}$$

Singlet



Non-Singlet



$$C_{i,\dots,j} \equiv \frac{i}{2}\int d\mathrm{LIPS}\Big[M^{\text{tree}}(\ell_1, i, i+1, \dots, j, \ell_2) \times$$
$$\times\, M^{\text{tree}}(-\ell_2, j+1, j+2, \dots, i-1, -\ell_1)\Big]$$

Amplitude computations and observables in general relativity

Idea

- Use of perturbative framework to compute observables in general relativity – vast literature. **(Blanchet talk)** **(Damour's talk)**
- Truncation to quantum computations to only classical terms.
- Many applications to computation post-Newtonian physics (PN) - NR-EFT methods/truncation to classical physics - Ideal for the low-velocity situations of planetary orbits, satellites, and large-distance effects
  - Hamiltonians for template generation
  - Radiation effects
  - Spin effects

Relativistic amplitudes to generate PM results

- **View-point:** relativistic scattering amplitudes (generally covariant theory of gravity coupled to matter)
- **Flat metric**. Full metric is treated perturbatively around Minkowskian background.
- **Post-Minkowskian expansion:** Keep all velocity terms in expansion while expanding order by order in Newton's constant
  - **Hamiltonians:** one and two-loops
  - **Scattering angle**: Amplitude -> GR map from Quantum Mechanical correspondence limit

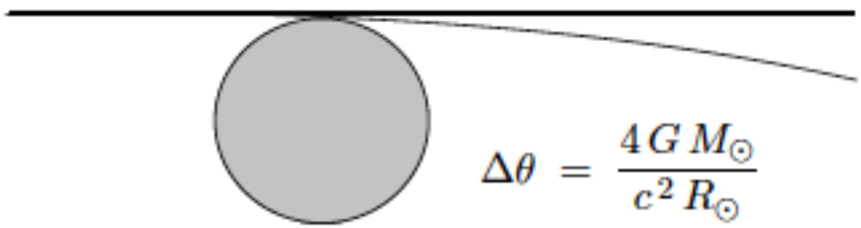
Relativistic amplitudes to generate PM results

- Some recent amplitude computations: (Guevara and Cachazo; Guevara; Damour; NEJB, Damgaard, Festuccia, Plante, Vanhove; Cheung, Rothstein, Solon; O'Connell, Maybee, Kosower; Collado, Di Vecchia, Russo)
- Scattering angle in post-Minkowskian formalism: (Westpfahl; Damour; Vines; NEJB, Damgaard, Festuccia, Plante, Vanhove)
- PM Hamiltonians: (Cheung, Rothstein, Solon; Bern, Cheung, Roiban, Shen, Solon, Zeng)

(See Damour's talk) (See Solon's talk)

Massless matter

- As an example we will consider scattering of massless matter



$$\Delta\theta = \frac{4\,G\,M_\odot}{c^2\,R_\odot}$$

- Bending of light/massless matter around the Sun
- New features: mass-less external fields ~> IR singularities
- New test of universality of matter

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

where

$$\mathcal{S}_{\text{scalar}} = \int d^4x \sqrt{-g} \left(-\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}\Big((\partial_\mu \Phi)^2 - M^2 \Phi^2 \Big) \right)$$

$$\mathcal{S}_{\text{fermion}} = \frac{i}{2} \int d^4x \sqrt{-g}\, \bar{\chi} \not{D} \chi \,,$$

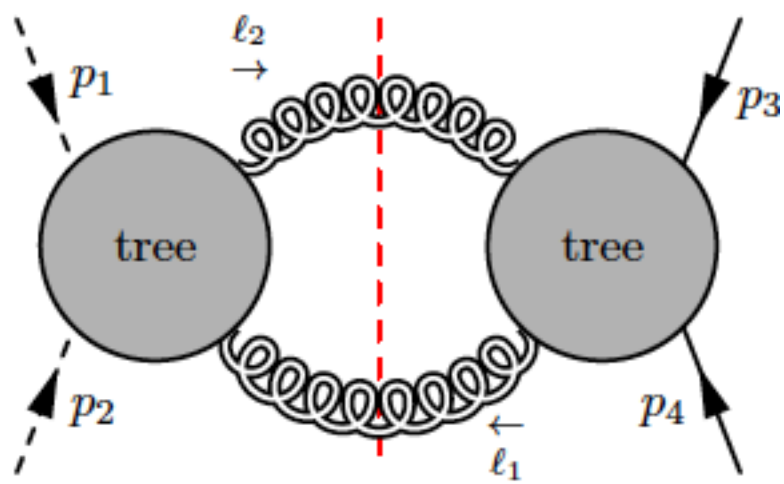
$$\mathcal{S}_{\text{QED}} = -\frac{1}{4} \int d^4x \sqrt{-g}\, (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2$$

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

We want to compute the cut



Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

We want to compute the cut

$$\mathcal{M}_X^{(2)}(p_1, p_2, p_3, p_4)\Big|_{\text{disc}} := \frac{1}{2!\, i} \mu^{2\epsilon} \int d\text{LIPS}(\ell_1, -\ell_2)\, (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \times \sum_{\lambda_1, \lambda_2} \mathcal{M}^{(1)}_{X^2 G^2}(p_1, \ell_1, p_2 - \ell_2) \times \mathcal{M}^{(1)}_{\phi^2 G^2}(p_3, \ell_2, p_4, -\ell_1)^\dagger$$

Photons and scalars

For photons we have

$$i\overset{0}{\mathcal{M}}^{[h^+(k_1)h^-(k_2)]}_{[\gamma^+(p_1)\gamma^-(p_2)]} = \frac{\kappa^2}{4}\frac{[p_1\,k_1]^2\,\langle p_2\,k_2\rangle^2\,\langle k_2|p_1|k_1]^2}{(p_1\cdot p_2)(p_1\cdot k_1)(p_1\cdot k_2)}$$

While for scalars

$$\begin{aligned} i\overset{0}{\mathcal{M}}^{[h^+(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\frac{M^4\,[k_1\,k_2]^4}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)} \\ i\overset{0}{\mathcal{M}}^{[h^-(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\frac{\langle k_1|p_1|k_2]^2\,\langle k_1|p_2|k_2]^2}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)} \end{aligned}$$

Super compact compared to Feynman diagram results

Result for the amplitude

We can rewrite

$$\mathcal{M}_{\varphi}^{(2)}(p_1,p_2,p_3,p_4) = -\frac{\kappa^4}{32t^2\, i}\sum_{i=1}^{2}\sum_{j=3}^{4}\int\frac{d^D\ell\,\mu^{2\epsilon}}{(2\pi)^D}\,\frac{\mathcal{N}^S}{\ell_1^2\ell_2^2(p_i\cdot\ell_1)(p_j\cdot\ell_1)}$$

where

$$\mathcal{N}^0_{\mathrm{non-singlet}} = \frac{1}{2}\left[\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)\right)^4 + \left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\right)^4\right]$$

Scalar case

$$\mathcal{N}^{\frac{1}{2}\,+-}_{\mathrm{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)^3\mathrm{tr}_+(p_1 p_3 p_2 \ell_1 p_3 \ell_2)\right) - (\ell_1 \leftrightarrow \ell_2)}{\langle p_2|p_3|p_1]}$$

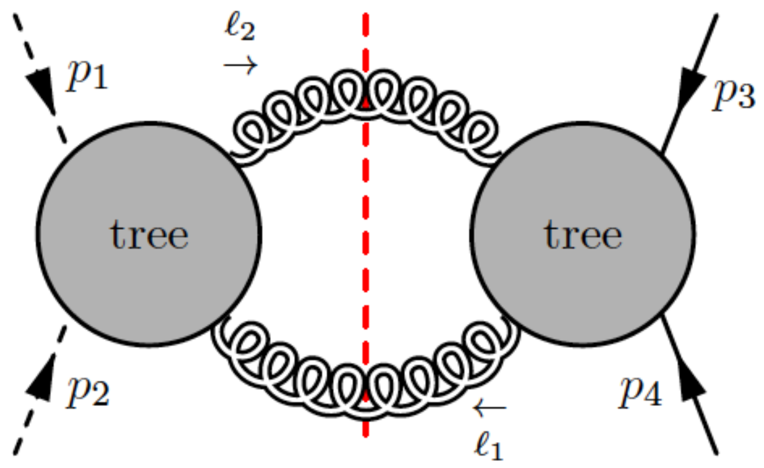
Fermion case

$$\mathcal{N}^{1\,+-}_{\mathrm{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\mathrm{tr}_+(\ell_2 p_3 \ell_1 p_1 p_3 p_2)\right)^2 + (\ell_1 \leftrightarrow \ell_2)}{\langle p_1|p_3|p_2]^2}$$

Photon case

Combine spinor expressions into traces

Result for the amplitude



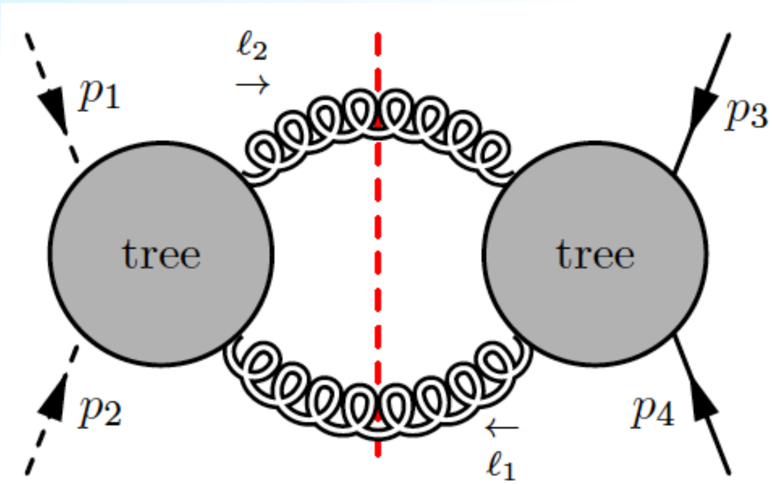
1) Expand out traces
2) Reduce to scalar basis of integrals
3) Isolate coefficients

(Bern, Dixon, Dunbar, Kosower)

$$bo^S(t,s)\, I_4(t,s) + bo^S(t,u)\, I_4(t,u)$$

$$+\, t^S_{12}(t)\, I_3(t,0) + t^S_{34}(t)\, I_3(t,M^2) + bu^S(t,0)\, I_2(t,0)$$

Result for the amplitude

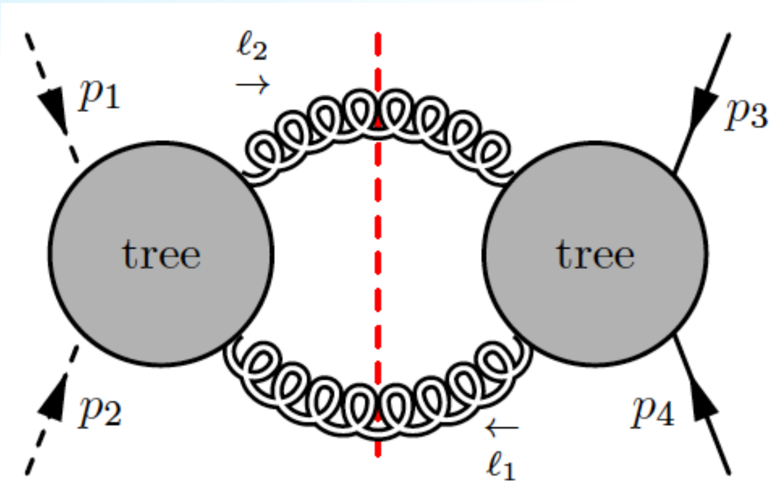


$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

$$-\frac{\mathcal{N}^X}{\hbar}\left[\hbar\frac{\kappa^4}{4}\Big(4(M\omega)^4(I_4(t,u)+I_4(t,s))+3(M\omega)^2 t I_3(t)\right.$$
$$\left. -15(M^2\omega)^2 I_3(t,M)+bu^X(M\omega)^2 I_2(t)\Big)\right]$$

Result for the amplitude



$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

Taking the post-Newtonian non-relativistic low energy limit

$$\frac{\mathcal{N}^X}{\hbar}(M\omega)^2\Big[-\kappa^4\frac{15}{512}\frac{M}{|\boldsymbol{q}|} - \hbar\kappa^4\frac{15}{512\pi^2}\log\left(\frac{\boldsymbol{q}^2}{M^2}\right) + \hbar\kappa^4\frac{bu^X}{(8\pi)^2}\log\left(\frac{\boldsymbol{q}^2}{\mu^2}\right)$$
$$-\hbar\kappa^4\frac{3}{128\pi^2}\log^2\left(\frac{\boldsymbol{q}^2}{\mu^2}\right) + \kappa^4\frac{M\omega}{8\pi}\frac{i}{\boldsymbol{q}^2}\log\left(\frac{\boldsymbol{q}^2}{M^2}\right)\Big]$$

(NEJB, Donoghue, Holstein, Plante, Vanhove)

Making connection to general relativity

General metric

$$ds^2 = A(r)dt^2 - B(r)^2 dr^2 - r^2 d\Omega^2$$

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2G_N M}{r}$$ Schwarzschild

$$\theta = \frac{4G_N M}{R} + \frac{4G_N^2 M^2}{R^2}\left(\frac{15\pi}{16} - 1\right) + \ldots$$

Can we reproduce?

Stationary phase method

We apply a Fourier transformation to impact parameter space and exponentiate into eikonal phases, so that a stationary phase method can be applied.

(See e.g. Akhoury, Saotome and Sterman)

$$\mathcal{M}(\boldsymbol{q}) = \mathcal{M}_1^{(1)}(\boldsymbol{q}) + \mathcal{M}^{(2)}(\boldsymbol{q})$$

$$\begin{aligned}\mathcal{M}(\boldsymbol{b}) &= 2(s - M^2)\left[(1 + i\chi_2)e^{i\chi_1} - 1\right] \\ &\simeq 2(s - M^2)\left[e^{i(\chi_1 + \chi_2)} - 1\right]\end{aligned}$$

Stationary phase method

Now we can compute

$$\chi_1(\boldsymbol{b}) = \frac{\kappa^2 M E}{4} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \frac{1}{\boldsymbol{q}^2}$$
$$\simeq 4 G_N M E \left[\frac{1}{d-2} - \log(b/2) - \gamma_E\right]$$

$$\chi_2(\boldsymbol{b}) = G_N^2 M^2 E \frac{15\pi}{4b} + \frac{G_N^2 M^2 E}{2\pi b^2}\left(8bu^\eta - 15 + 48\log\frac{2b_0}{b}\right)$$

Stationary phase method

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\,b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

We arrive at:

$$2\sin\frac{\theta}{2} \simeq \theta = -\frac{1}{E}\frac{\partial}{\partial b}\left(\chi_1(b) + \chi_2(b)\right)$$

Stationary phase method

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\,b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

Or:

$$\theta \simeq \frac{4G_N M}{b} + \frac{15}{4}\frac{G_N^2 M^2 \pi}{b^2} + \left(8bu^S + 9 - 48\log\frac{b}{2b_0}\right)\frac{\hbar G_N^2 M}{\pi b^3} + \ldots$$

Bending of light

Interpreted as a bending angle (eikonal approximation) we have:

$$\theta_\eta \simeq \frac{4GM}{b} + \frac{15}{4}\frac{G^2M^2\pi}{b^2}$$

plus a quantum effect of the order of magnitude:

$$+\frac{8bu^\eta + 9 + 48\log\frac{b}{2r_o}}{\pi}\frac{G^2\hbar M}{b^3}$$

We see that we have universality between scalars, fermions and photons only for the 'Newton' and 'post-Newtonian' contributions

(NEJB, Donoghue, Holstein, Plante, Vanhove; Bai, Huang; Collado, Di Vecchia, Russo, Thomas; Chi)

Massive scattering: Scalar interaction potentials (tree)

Tree level

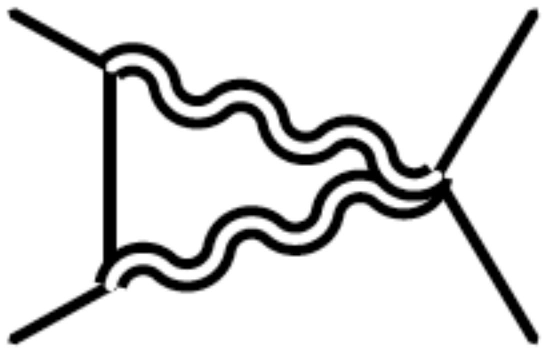
$\mathcal{M}_1 =$

$$= -\frac{16\pi G}{q^2}\left(m_1^2 m_2^2 - 2(p_1 \cdot p_4)^2 - (p_1 \cdot p_4)q^2\right)$$

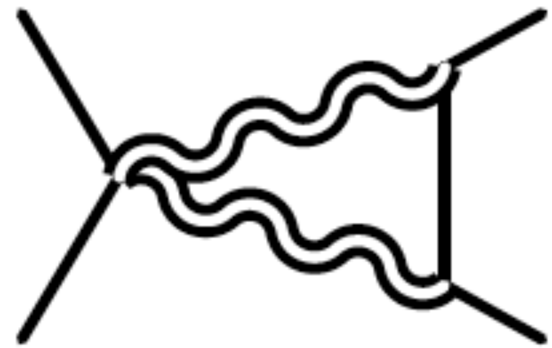
Massive scattering: Scalar interaction potentials (one-loop)

One-loop level

$\mathcal{M}_2 =$



$+$



$$= -i(8\pi G)^2 \left(\frac{c(m_1, m_2) I_{\triangleright}(p_1, q)}{(q^2 - 4m_1^2)^2} + \frac{c(m_2, m_1) I_{\triangleright}(p_4, -q)}{(q^2 - 4m_2^2)^2} \right)$$

Classical contribution from one-loop amplitude

General relativity encoded in triangle coefficients

$$
\begin{aligned}
c(m_1, m_2) = (q^2)^5 + (q^2)^4 \left(6 p_1 \cdot p_4 - 10 m_1^2\right) \\
+ (q^2)^3 \left(12 (p_1 \cdot p_4)^2 - 60 m_1^2 p_1 \cdot p_4 - 2 m_1^2 m_2^2 + 30 m_1^4\right) \\
-(q^2)^2 \left(120 m_1^2 (p_1 \cdot p_4)^2 - 180 m_1^4 p_1 \cdot p_4 - 20 m_1^4 m_2^2 + 20 m_1^6\right) \\
+q^2 \left(360 m_1^4 (p_1 \cdot p_4)^4 - 120 m_1^6 p_1 \cdot p_4 - 4 m_1^6 (m_1^2 + 15 m_2^2)\right) \\
+ 48 m_1^8 m_2^2 - 240 m_1^6 (p_1 \cdot p_4)^2
\end{aligned}
$$

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Post-Newtonian potentials

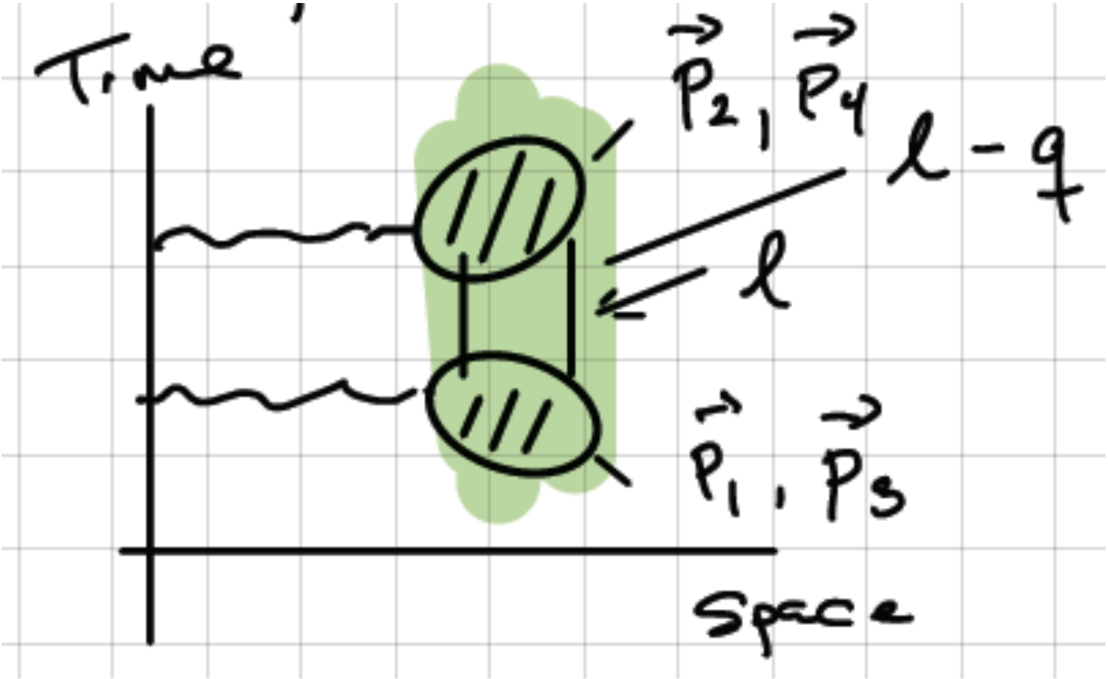
Leading order in q

$$\mathcal{M}_2 = \frac{6\pi^2 G^2}{|\vec{q}|}(m_1 + m_2)(5(p_1 \cdot p_4)^2 - m_1^2 m_2^2)$$

All momenta provided at infinity, contractions are done using flat space metric (Minkowski), no reference to coordinates. Gauge invariant expression – to derive potential we have to introduce coordinates, Fourier transform and expand subleading terms in q^0.

Born subtraction

- Born subtraction important to make contact with Post-Newtonian limit.



Limit: Post-Newtonian interaction potential

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_4^2}{2m_2} - \frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_4^4}{8m_2^3}$$
$$- \frac{Gm_1m_2}{r} - \frac{G^2m_1m_2(m_1+m_2)}{2r^2}$$
$$- \frac{Gm_1m_2}{2r}\left(\frac{3\vec{p}_1^2}{m_1^2} + \frac{3\vec{p}_4^2}{m_2^2} - \frac{7\vec{p}_1\cdot\vec{p}_4}{m_1m_2} - \frac{(\vec{p}_1\cdot\vec{r})(\vec{p}_4\cdot\vec{r})}{m_1m_2r^2}\right)$$

(Einstein-Infeld-Hoffman)

Subtraction of Post-Newtonian tree-level Born term to in order to get the correct potential (3 – 7/2 -> -1/2)

Post-Minkowskian expansion

Will use similar eikonal setup as for bending of light (extended to massive case):

$$\vec{p}_1 = -\vec{p}_4$$

b orthogonal and

$$b \equiv |\vec{b}|$$

Amplitude computed

$$M(\vec{b}) \equiv \int d^2\vec{q}e^{-i\vec{q}\cdot\vec{b}}M(\vec{q})$$

$$M(\vec{b}) = 4p(E_1 + E_2)(e^{i\chi(\vec{b})} - 1)$$

Eikonal phase

Post-Minkowskian expansion

Stationary phase condition (leading order in q)

$$2\sin(\theta/2) = \frac{-2M}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b}\left(\chi_1(b) + \chi_2(b)\right)$$

$$\chi_1(b) = 2G\,\frac{\hat{M}^4 - 2m_1^2 m_2^2}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}}\left(\frac{1}{d-2} - \log\left(\frac{b}{2}\right) - \gamma_E\right)$$

$$\chi_2(b) = \frac{3\pi G^2}{8\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}}\frac{m_1 + m_2}{b}(5\hat{M}^4 - 4m_1^2 m_2^2)$$

Post-Minkowskian expansion

Final result becomes

$$2\sin\left(\frac{\theta}{2}\right) = \frac{4GM}{b}\Big(\frac{\hat{M}^4 - 2m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2}$$

$$+ \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{M}^4 - 4m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2}\Big)$$

Agrees with
(Westpfahl, Damour)

Light-like limit

$$\theta = \frac{4Gm_1}{b} + \frac{15\pi}{4}\frac{G^2m_1^2}{b^2}$$

Post-Minkowskian expansion

- So we have made contact with general relativity without making *a direct* reference to a Post-Minkowskian Hamiltonian.
- Can we *always* avoid reference to Hamiltonian? (needs to be investigated further..)
- Understanding of one-loop scattering angle results from Post-Minkowskian Hamilton? Proper Born subtraction yields equivalence of results!
- Still need for better understanding at higher loops...

Outlook

How define Amplitude -> GR map:

Non-trivial problem: We have illustrated some examples of such a map that contains interesting physics. This provides a direct way to derive the one-loop PM bending angle in general relativity.

However: Good prospects for further theoretical and practical breakthroughs.

- Spin effects (gravity phenomenology)
- Radiation (higher precision and better templates)
- Higher precision/loops

Outlook

Already **extensive work** on QFT approach to gravity. **(Damour's talk)**

Forms the theoretical backbone of current investigations using templates at LIGO/Virgo.

Observation: Amplitude angle provides *new efficiency* to calculations.

1) Double-copy allows recycling of Yang-Mills results in gravity.

2) Off-shell -> On-shell (removes clutter from computations, while essence contributions remains). *However important considerations when throwing away off-shell information.* (coordinate dependence etc.)

3) Classical parts are possible to identify and target independently of quantum contributions. (non-analytic pieces, have unique cuts)

Hope: That reconsidering how to do such computations can be used to increase precision for gravity observables.

Exciting times ahead!!

New on-shell toolbox for gravity computations

New techniques for computation of physical observables in general relativity