Workshop on Multi-Loop Calculations: Methods and Applications

campus de Jussieu

THE FOKKER ACTION OF COMPACT BINARY SYSTEMS
at
THE FOURTH POST-NEWTONIAN ORDER

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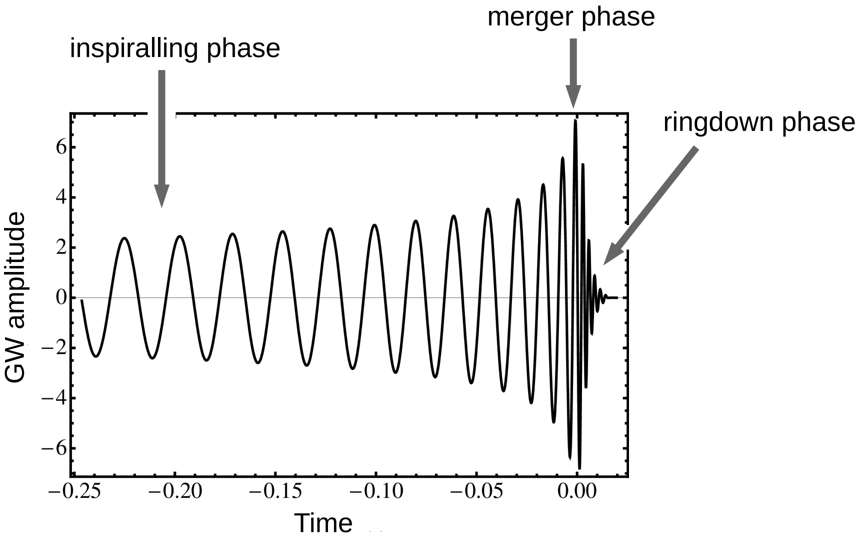
The gravitational chirp of binary black holes



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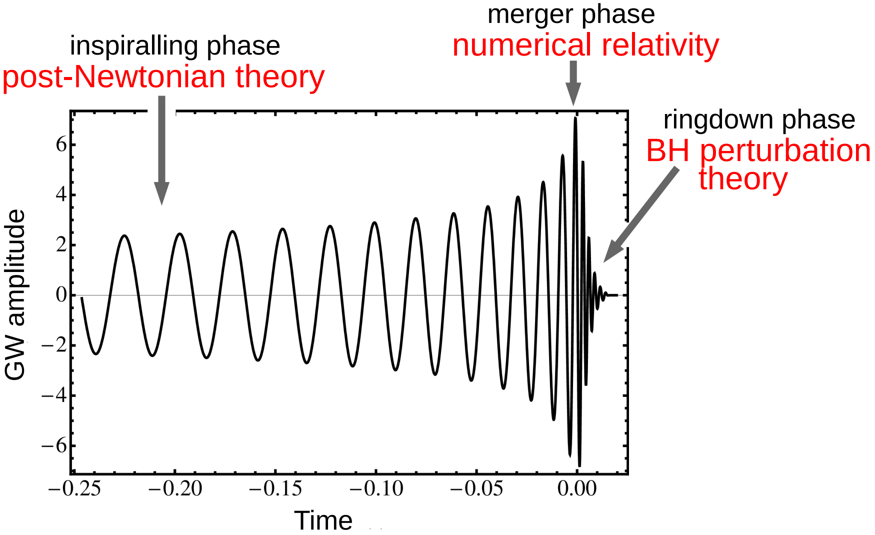

The GW templates of compact binaries

1. In principle, the templates are obtained by matching together:
   - A high-order 3.5PN waveform for the inspiral [Blanchet *et al.* 1998, 2002, 2004]
   - A highly accurate numerical waveform for the merger and ringdown [Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006]
2. In the practical data analysis, for black hole binaries (such as GW150914), effective methods that interpolate between the PN and NR play a key role:
   - Hybrid inspiral-merger-ringdown (IMR) waveforms [Ajith *et al.* 2011] are constructed by matching the PN and NR waveforms in a time interval through an intermediate phenomenological phase
   - Effective-one-body (EOB) waveforms [Buonanno & Damour 1998] are based on resummation techniques extending the domain of validity of the PN approximation beyond the inspiral phase
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Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu}\dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu}\dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)$$

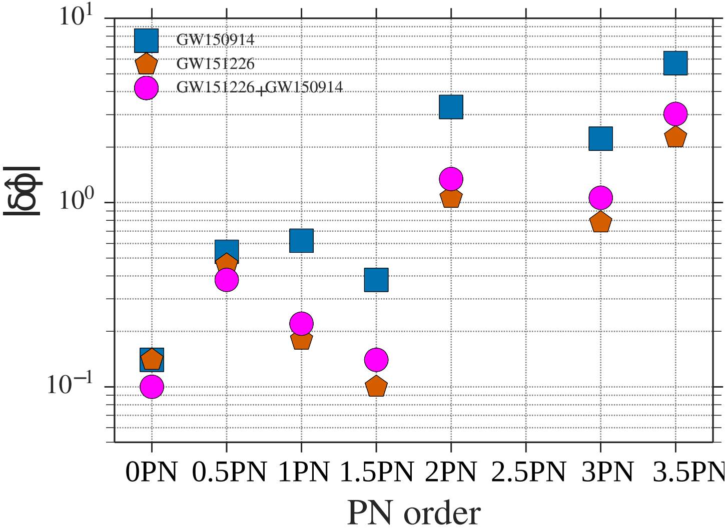
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Measurement of PN parameters from BH events

[LIGO/Virgo collaboration 2016]



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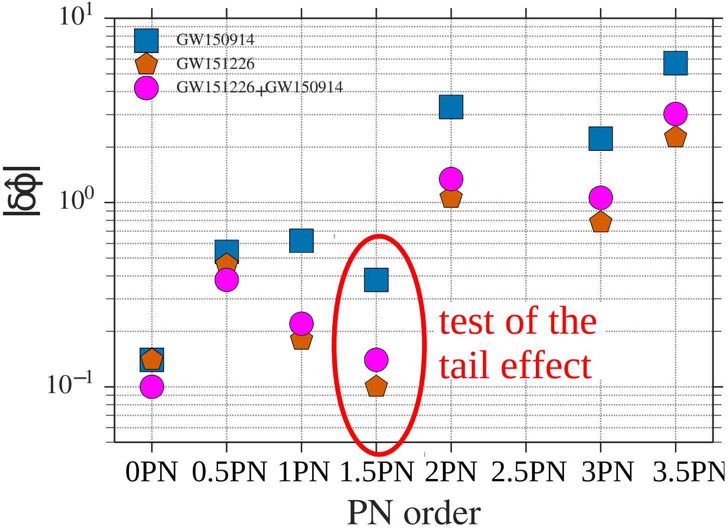

Measurement of PN parameters from the NS event

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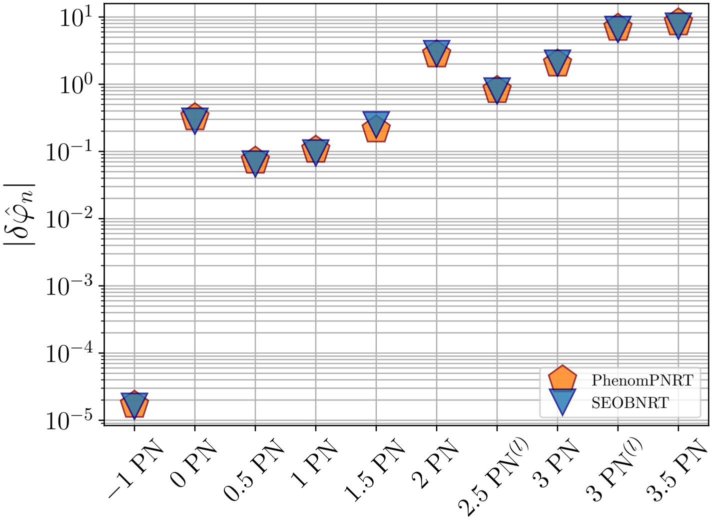

Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN[1] 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	4PN 2.5PN (NL) SO 4PN (NNL) SS	2PN 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS	2PN 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN non-spin		

- Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
- SO effects are known in radiation field up to 4PN
- SS in radiation field known to 3PN

[1] The 4.5PN coefficient is also known [Marchand, Blanchet & Faye 2016]

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Wave-generation and radiation-reaction problems

- Gauge-fixed Einstein field equations

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(harmonic gauge condition)}$$

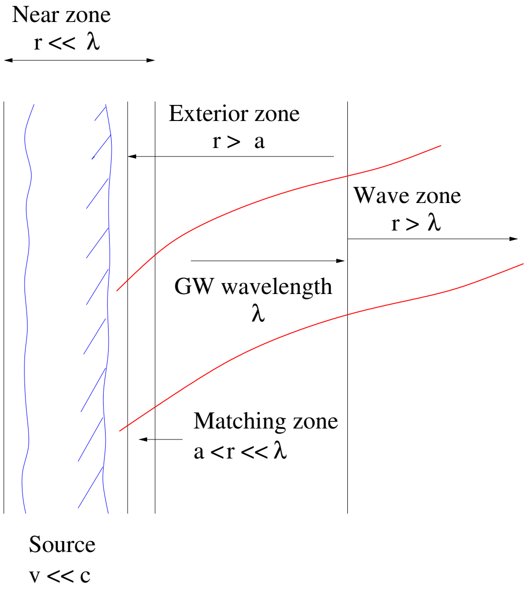
$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \qquad \text{(flat space-time wave equation)}$$

- Pseudo-tensor of matter and gravitational fields

$$\boxed{\tau^{\mu\nu} = \underbrace{|g|T^{\mu\nu}}_{\text{matter part}} + \frac{c^4}{16\pi G}\underbrace{\Lambda^{\mu\nu}(h,\partial h,\partial^2 h)}_{\text{gravitational part}}}$$

1. Radiation-reaction problem: solve the EFE inside the compact-support source to get the reaction forces acting the an isolated source
2. Wave generation problem: solve the EFE in vacuum outside the source (including the regions at infinity) to get the waveform as a functional of the source parameters

Regions of space around the GW source



Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Look for the general multipolar expansion $\mathcal{M}(h)$ generated outside the source in the form [Bonnor 1959, Bonnor & Rotenberg 1961]

$$\mathcal{M}(h) = \underbrace{G\, h_{(1)} + G^2\, h_{(1)} + \cdots + G^n\, h_{(n)} + \cdots}_{\text{formal post-Minkowskian expansion}}$$

2. Start from the general multipolar solution of the vacuum field equation at linear order [Thorne 1980]

$$h_{(1)}[M_L, S_L] = \sum_{\ell=0}^{+\infty} \left[\partial_L \left(\frac{1}{r} \underbrace{M_L(t-r/c)}_{\text{mass-type moment}} \right) + \varepsilon_{abc}\, \partial_L \left(\frac{1}{r} \underbrace{S_L(t-r/c)}_{\text{current-type moment}} \right) \right]$$

3. Iterate that solution using a specific regularization scheme based on A.C. in $B \in \mathbb{C}$ to cope with the singularity of the multipole expansion when $r \to 0$

$$\underset{B=0}{\text{Finite Part}}\, \Box^{-1}_{\text{ret}} \left[(r/r_0)^B f \right]$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. At n-th post-Minkowskian order we need to solve

$$\partial_\nu h^{\mu\nu}_{(n)} = 0$$
$$\Box h^{\mu\nu}_{(n)} = \Lambda^{\mu\nu}\big(\underbrace{h_{(1)},\cdots h_{(n-1)}}_{\text{known from previous iterations}}\big)$$

2. A particular solution with the required multipole structure reads

$$u^{\mu\nu}_{(n)} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{Ret}}\left[\left(\frac{r}{r_0}\right)^B \Lambda^{\mu\nu}_{(n)}\right]$$

3. In order to guarantee that the harmonic gauge condition $\partial_\nu h^{\mu\nu}_{(n)} = 0$ is satisfied we add an homogeneous solution $v^{\mu\nu}_{(n)}$ hence

$$\boxed{h^{\mu\nu}_{(n)} = u^{\mu\nu}_{(n)} + v^{\mu\nu}_{(n)}}$$

4. The MPM solution is generated by a plug-and-grind algorithm up to any PM order as a functional of the multipole moments $M_L(t)$ and $S_L(t)$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988 1992; Blanchet 1987, 1993, 1998]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

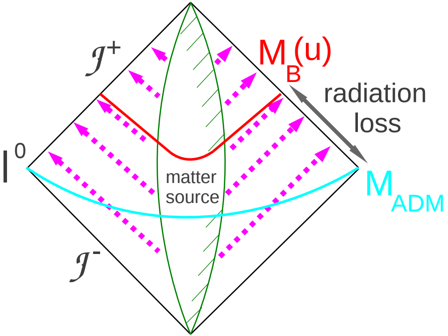
Theorem 2:
The general structure of the PN expansion is

$$h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)$$

Theorem 3:
The MPM solution is asymptotically flat at future null infinity in the sense of Penrose and recovers the Bondi-Sachs formalism [1960s]

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]



$$M_{\mathsf{B}}(u) = M_{\mathsf{ADM}} - \overbrace{\frac{G}{5c^7}\int_{-\infty}^{u} \mathrm{d}t\, M_{ij}^{(3)}(t) M_{ij}^{(3)}(t)}^{\text{mass-energy emitted in GW}}$$

$$+ \begin{cases} \text{higher-order multipole moments and} \\ \text{higher-order PM approximations} \end{cases}$$

Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. Most general multipolar(-post-Minkowskian) solution in the source's exterior

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{ret}}\left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda)\right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{M_L(t-r/c)}{r} \right\}$$

where the homogeneous solution is parametrized by multipole moments

2. Most general PN solution in the source's near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B \bar{\tau}\right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{A_L(t-r/c) - A_L(t+r/c)}{r} \right\}$$

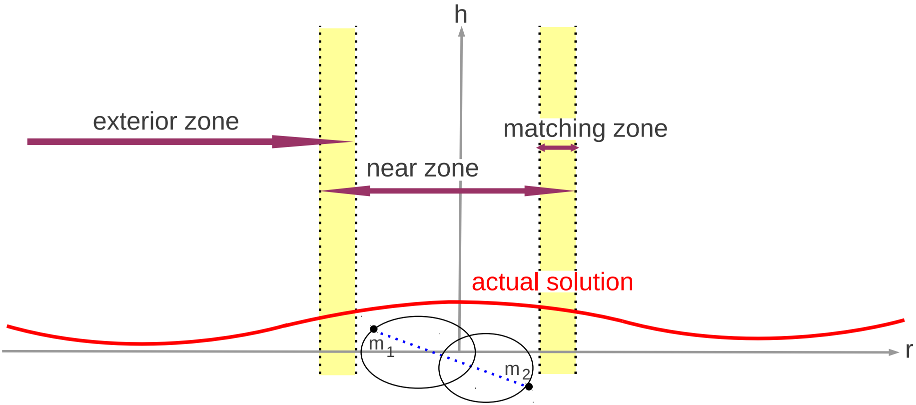
where the homogeneous solution (regular when $r \to 0$) is parametrized by "radiation reaction" multipole moments

3. We impose the matching equation

$$\boxed{\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})}$$

Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]



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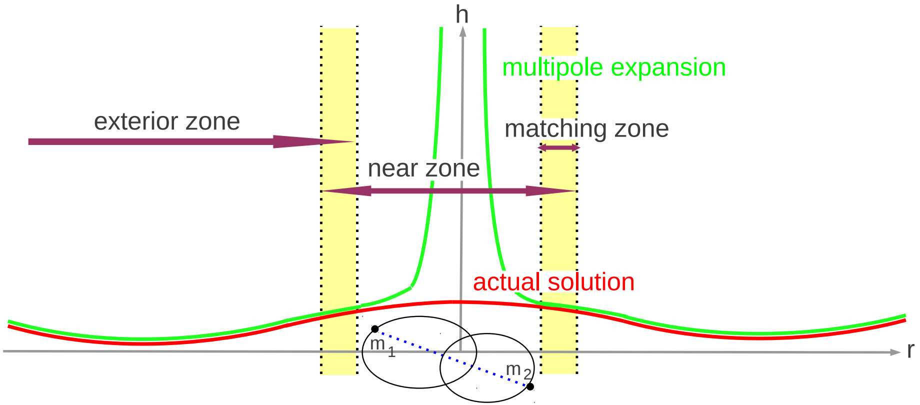

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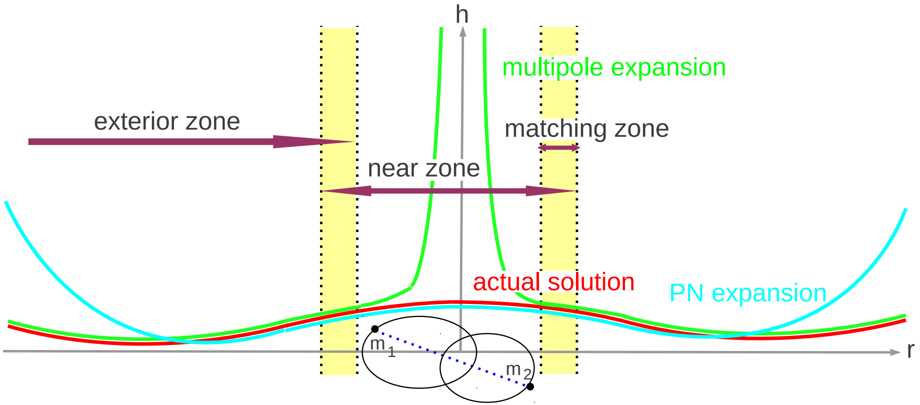

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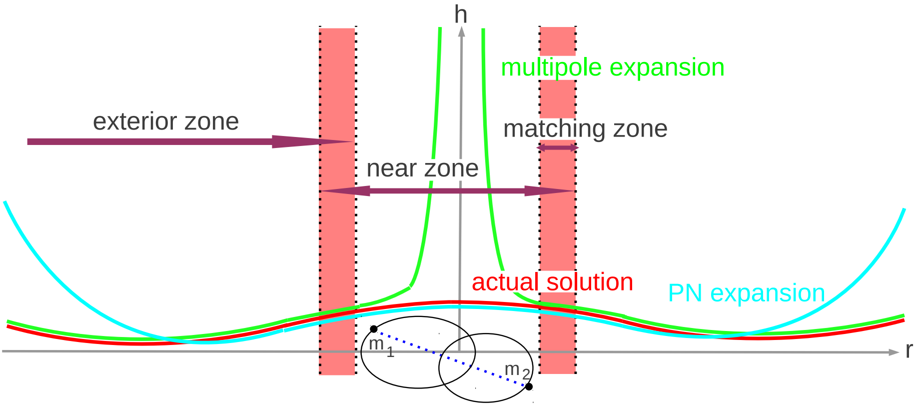


Near zone expansion of the multipole expansion

Lemma 1: [Blanchet 1993, 1998]

$$\overline{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right]} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{sym}} \left[\left(\frac{r}{r_0}\right)^B \overline{\mathcal{M}(\Lambda)} \right]$$

$$- \frac{4G}{c^4} \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t - r/c) - \mathcal{R}_L(t + r/c)}{2r} \right\}}_{\text{antisymmetric type homogeneous solution}}$$

where the radiation reaction multipole moments are

$$\boxed{\mathcal{R}_L(u) = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \hat{x}_L \int_1^{+\infty} \mathrm{d}z \, \gamma_\ell(z) \underbrace{\mathcal{M}(\tau)(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}}$$

The finite part at $B = 0$ plays the role of an UV regularization $(r \to 0)$

Far zone expansion of the PN expansion

Lemma 2: [Poujade & Blanchet 2001]

$$\mathcal{M}\left(\mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\bar{\tau}\right]\right) = \mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\mathcal{M}(\bar{\tau})\right] - \underbrace{\frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{F}_L(t-r/c)+\mathcal{F}_L(t+r/c)}{2r}\right\}}_{\text{symmetric type homogeneous solution}}$$

$$\boxed{\mathcal{F}_L(u) = \mathop{\mathrm{FP}}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an IR regularization ($r\to+\infty$)

General solution of the matching equation

1. In the far zone

$$\mathcal{M}(h) = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{F}_L(t - r/c)}{r} \right\}}_{\text{source's multipole moments}}$$

2. In the near zone [Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

$$\bar{h} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] - \underbrace{\overline{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t - r/c) - \mathcal{R}_L(t + r/c)}{r} \right\}}}_{\text{non-local tail term (4PN order)}}$$

PROBLEM OF THE MOTION

4PN equations of motion of compact binary systems

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory

The Fokker Lagrangian approach to the 4PN EOM

Based on collaborations with





Laura Bernard, **Alejandro Bohé**, **Guillaume Faye**,
Tanguy Marchand & **Sylvain Marsat**

[PRD **93**, 084037 (2016); **95**, 044026 (2017); **96**, 104043 (2017); **97**, 044023 (2018); **97**, 044037 (2018)]

Fokker action of N point particles

- Gauge-fixed Einstein-Hilbert action of N point particles

$$S_{\mathsf{EH}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}\Gamma^\mu\Gamma_\mu}_{\text{gauge-fixing term}}\Big] - \sum_a \underbrace{m_a c^2 \int \mathrm{d}\tau_a}_{N\text{ point particles}}$$

- The Fokker PN action is obtained by inserting an explicit iterated PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{x}_a(t),\boldsymbol{v}_a(t),\cdots)$$

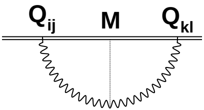
- The PN equations of motion of the N particles (self-gravitating system) are

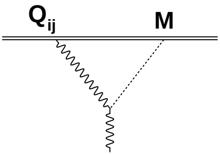
$$\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{x}_a} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{x}_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_a}\right) + \cdots = 0$$

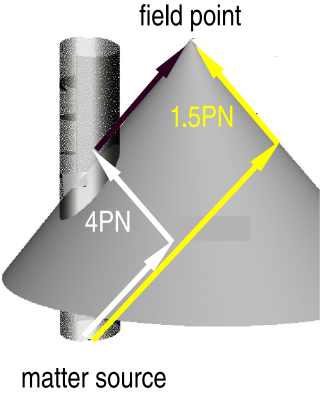
- The Fokker action is equivalent to the effective action of the EFT

The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley, Leibovich, Porto & Ross 2016]







- In the near zone (4PN effect)

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \,\text{Pf} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} Q^{(3)}_{ij}(t)\, Q^{(3)}_{ij}(t')$$

- In the far zone (1.5PN effect)

$$h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t'\, Q^{(4)}_{ij}(t') \ln\left(\frac{t-t'}{\tau_0}\right)$$

Non-locality in time of the 4PN dynamics

1. Because of the tail effect at 4PN order the Lagrangian or Hamiltonian becomes non-local in time

$$H\left[\mathbf{x},\mathbf{p}\right] = H_0\left(\mathbf{x},\mathbf{p}\right) + \underbrace{H_{\text{tail}}\left[\mathbf{x},\mathbf{p}\right]}_{\text{non-local piece at 4PN}}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. The conserved energy is not given by the Hamiltonian on-shell but $E = H + \Delta H$ where the AC part of the correction term averages to zero and

$$\boxed{\Delta H^{\text{DC}} = -\frac{2GM}{c^3}\mathcal{F}^{\text{GW}} = -\frac{2G^2M}{5c^8}\left\langle\left(Q^{(3)}_{ij}\right)^2\right\rangle}$$

4. On the other hand [DJS] perform a non-local shift to transform the Hamiltonian into a local one, and both procedures are equivalent

Potential modes versus radiation modes

- The potential modes are responsible for conservative near zone effect and can be computed with the symmetric propagator (when neglecting radiation reaction effects)
- The radiation modes are conservative effects coming from gravitational waves propagating at infinity and re-expanded in the near zone. The first radiation effect in the Fokker action is the non local tail effect at 4PN order
- To high PN order there is a complicated mix up between potential and radiation modes encapsuled in the general formula

$$\bar{h} = \underbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right]}_{\text{potential modes}} - \underbrace{\frac{4G}{c^4} \overline{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t - r/c) - \mathcal{R}_L(t + r/c)}{r} \right\}}}_{\text{radiation modes}}$$

Dimensional regularization of the Fokker action

- UV divergences due to the modelling of compact objects by point particles plague the potential modes starting from the 3PN order
- IR divergences in the Einstein-Hilbert part of the Fokker action (potential modes) occur at the 4PN order
- The IR pole in the potential modes should be compensated by an UV pole coming from the non-local tail term at 4PN order (radiation mode)
- UV and IR divergences are treated with dimensional regularization ($d = 3 + \varepsilon$)

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi}\frac{\theta(t-r)}{r^{d-1}}\,\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right)$$

$$\gamma_s(z) = \frac{2\sqrt{\pi}}{\Gamma(s+1)\Gamma(-s-\frac{1}{2})}\left(z^2-1\right)^s \quad \text{(such that } \int_1^{+\infty} \mathrm{d}z\,\gamma_s(z) = 1\text{)}$$

- The regularization is followed by a renormalization in the form of shifts (or contact transformations) of the particles' world-lines

Potential mode contribution to IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\mathsf{HR}} = \underset{B=0}{\mathrm{FP}} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\mathsf{DR}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)$$

UV divergences coming from the radiation mode

1. At 4PN order the radiation mode is due to the presence of the tail effect described in 3 dimensions by

$$\mathcal{H}_{\text{tail}} = -\frac{4G}{c^4} \overline{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}$$

2. In d dimensions it reads

$$\mathcal{H}_{\text{tail}} = \sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \Delta^{-k} \hat{x}_L \, f_L^{(2k)}(t)$$

$$f_L(t) = \frac{(-)^{\ell+1}\tilde{k}}{4\pi\ell!} \mathop{\text{FP}}_{B=0} \int_1^{+\infty} \mathrm{d}z \, \gamma_{\frac{1-d}{2}}(z) \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B \hat{\partial}_L \left[\frac{\mathcal{M}(\Lambda)(\mathbf{y}, t - zr/c)}{r^{d-2}} \right]_{\mathbf{y}=\mathbf{x}}$$

3. In intermediate calculations of radiation modes it is important to keep the parameter B and apply first the limit $B \to 0$ for any $\varepsilon > 0$

The B-ε regularization scheme

1. Specializing to the quadratic mass quadrupole interaction $F_L \sim M \times Q_{ij}$ the multipolar source term will itself be of the type

$$\mathcal{M}(\Lambda)_L(r, t - zr/c) = r^{-2d+6-k} \int_1^{+\infty} \mathrm{d}y\, y^p\, \gamma_{\frac{1-d}{2}}(y) F_L(t - (y+z)r/c)$$

2. After a series of transformations we end up with

$$f_L = \underset{B=0}{\mathrm{FP}} \frac{(-)^{\ell+k}\, C_\ell^{p,k}(\varepsilon, B)}{2\ell + 1 + \varepsilon} \frac{\Gamma(2\varepsilon - B)}{\Gamma(\ell + k - 1 + 2\varepsilon - B)} \int_0^{+\infty} \mathrm{d}\tau\, \tau^{B-2\varepsilon}\, F_L^{(\ell+k-1)}(t-\tau)$$

3. The numerical coefficient is defined by analytic continuation in B and ε

$$C_\ell^{p,k}(\varepsilon, B) = \int_1^{+\infty} \mathrm{d}y\, y^p\, \gamma_{-1-\frac{\varepsilon}{2}}(y) \int_1^{+\infty} \mathrm{d}z\, (y+z)^{\ell+k-2+2\varepsilon-B} \gamma_{-\ell-1-\frac{\varepsilon}{2}}(z)$$

4. The regulator B is needed to protect against the divergence of this integral at infinity (when $y, z \to +\infty$, with $y \sim z$)

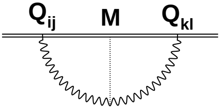
Ambiguity-free completion of the 4PN EOM

1. From the metric we obtain the equations of motion and then identify the corresponding gauge invariant term in the Fokker action
2. We find that the limit $B \to 0$ is finite (no poles) and we obtain a simple closed-form expression for the tail term in an arbitrary d dimension

$$S_{\mathrm{F}}^{\text{tail}} = K_d \,\frac{G^2 M}{c^8} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|^{2d-5}} Q_{ij}^{(3)}(t)\,Q_{ij}^{(3)}(t')$$

$$\text{with} \quad K_d = \frac{12-12d+5d^2-4d^3+d^4}{8(d-1)^2(d+2)}\left(\frac{2\ell_0^2}{\pi}\right)^{d-3}\frac{\Gamma(-\frac{d}{2})}{\Gamma(\frac{7}{2}-d)\Gamma(\frac{5}{2}-\frac{d}{2})}$$

3. This should correspond exactly to the (real-space version of the) Feynman diagram computed in Fourier space by the EFT community [Galley, Leibovich, Porto & Ross 2016]



Ambiguity-free completion of the 4PN EOM

1. In the limit $\varepsilon \to 0$ this gives Hadamard's "Partie finie" (Pf) integral

$$S_{\mathsf{F}}^{\text{tail}} = \frac{G^2 M}{5c^8} \, \underset{\tau_0}{\mathsf{Pf}} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|} Q_{ij}^{(3)}(t)\, Q_{ij}^{(3)}(t')$$

$$\text{with} \quad \tau_0 = \frac{\ell_0}{c\sqrt{\pi}} \exp\Big[\underbrace{\frac{1}{2\varepsilon}}_{\text{UV type pole}} - \frac{1}{2}\gamma_{\mathsf{E}} - \frac{41}{60} \Big]$$

2. We find that the UV pole exactly cancels the IR pole coming from the potential (Einstein-Hilbert) part of the Fokker action
3. Adding up all contributions we obtain the complete EOM at 4PN order with self-consistent derivation of previously conjectured "ambiguity" parameters
4. Recently the EFT approach has also succeeded in a full self-consistent ambiguity-free derivation of the 4PN EOM [Foffa & Sturani 2019; Foffa, Porto, Rothstein & Sturani 2019]
5. The three methods (ADM Hamiltonian, Fokker Lagrangian, EFT) are in perfect agreement on the final result