

Multi-loops in Classical Gravity

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Workshop on Multi-loop Calculations: Methods and Applications

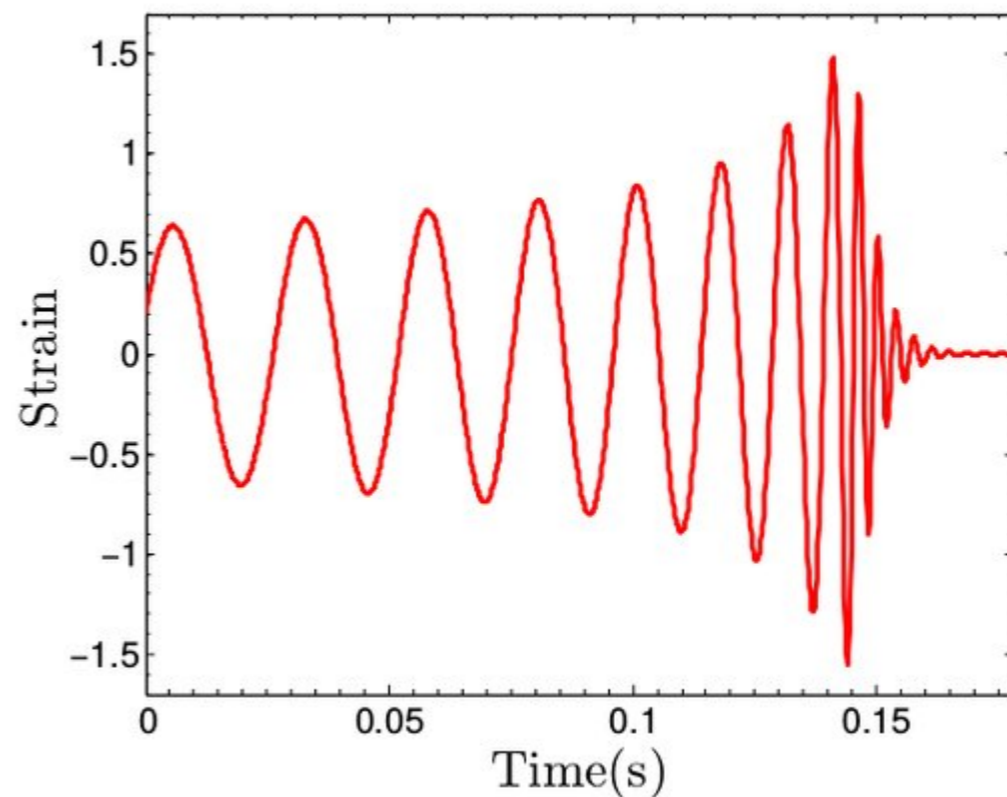
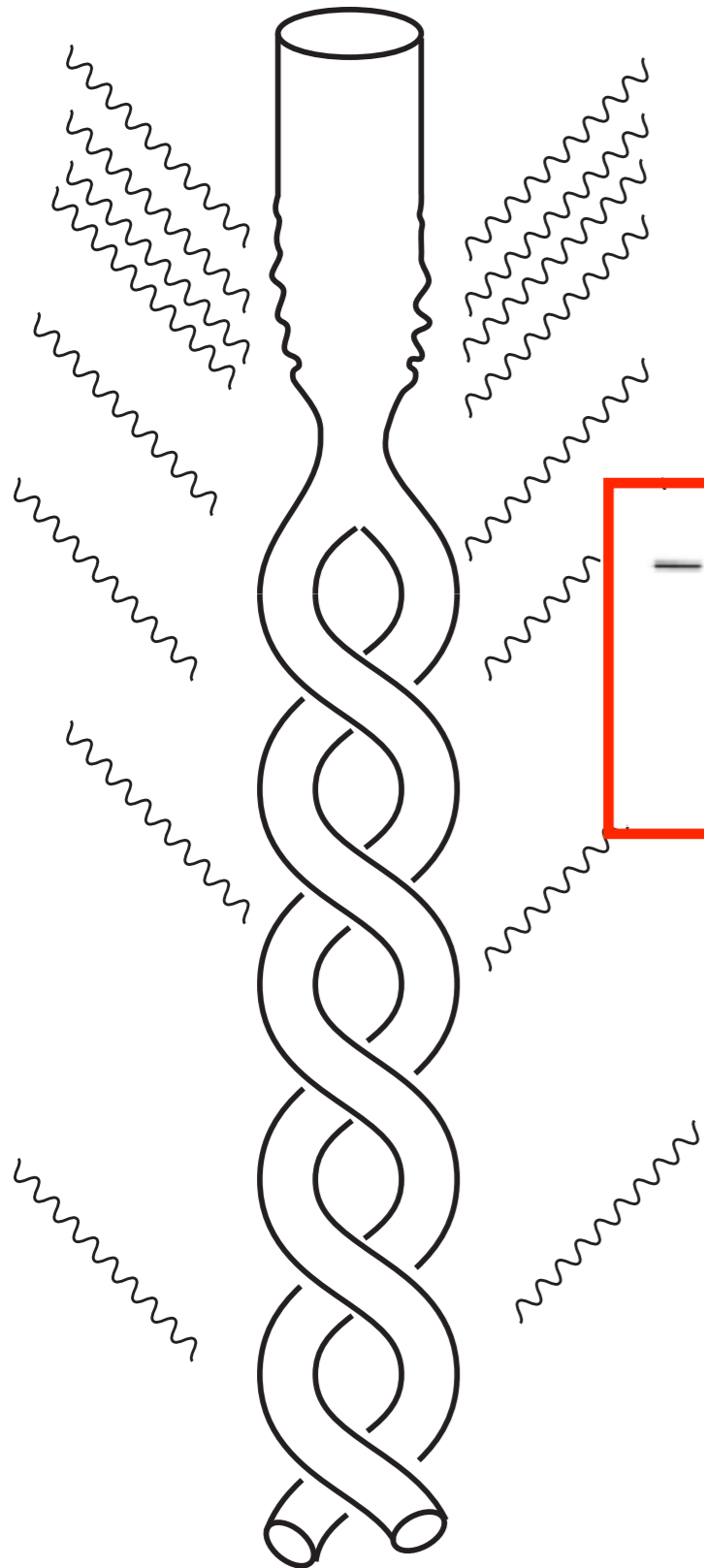
Sorbonne University, Paris, 14-15 May 2019

$$ds^2 = g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} = 0$$

$$-g^{\mu\nu} g_{\alpha\beta, \mu\nu} + g^{\mu\nu} g^{\rho\sigma} (g_{\alpha\mu, \rho} g_{\beta\nu, \sigma} - g_{\alpha\mu, \rho} g_{\beta\sigma, \nu} + g_{\alpha\mu, \rho} g_{\nu\sigma, \beta} + g_{\beta\mu, \rho} g_{\nu\sigma, \alpha} - \frac{1}{2} g_{\mu\rho, \alpha} g_{\nu\sigma, \beta}) = 0$$



Basic Physico-Mathematical Tools Having Allowed one to Predict GW signals from Coalescing Binary Black Holes

Matched Asymptotic Expansion approach to the motion of strongly self-gravitating bodies \rightarrow UV finiteness up to 5PN=5 loops

Post-Newtonian (PN) approximation theory to the motion of binary systems \rightarrow known up to 4 PN=4 loops

Post-Minkowskian (PM) approximation theory to the motion of binary systems \rightarrow known to 2 PM= $O(G^2)$ =1 loop; recent claim at 3PM= 2 loops [Bern-Cheung-Roiban-Shen-Solon-Zeng'19]

PN-matched Multipolar post-Minkowskian (MPM) approximation theory to the GW emission of binary systems \rightarrow known up to 3 loops

Effective One-Body (EOB) Approach to coalescing binary black holes (and binary neutron stars) \rightarrow known up to 4PN; and recently extended to 5PN = 5 loops [Bini-Damour-Geralico'19]

Gravitational Self-Force (SF) \rightarrow compute some terms up to 22 loops

Numerical Relativity (NR) simulations of coalescing binary black holes (and binary neutron stars)

MATCHED FILTERING SEARCH AND DATA ANALYSIS

O1: precomputed bank of $\sim 200\,000$ EOB templates for inspiralling and coalescing BBH GW waveforms: $m_1, m_2, \chi_1=S_1/m_1^2, \chi_2=S_2/m_2^2$ for $m_1+m_2 > 4M_{\text{sun}}$; + $\sim 50\,000$ PN inspiralling templates for $m_1+m_2 < 4M_{\text{sun}}$;
O2: $\sim 325\,000$ EOB templates + $75\,000$ PN templates

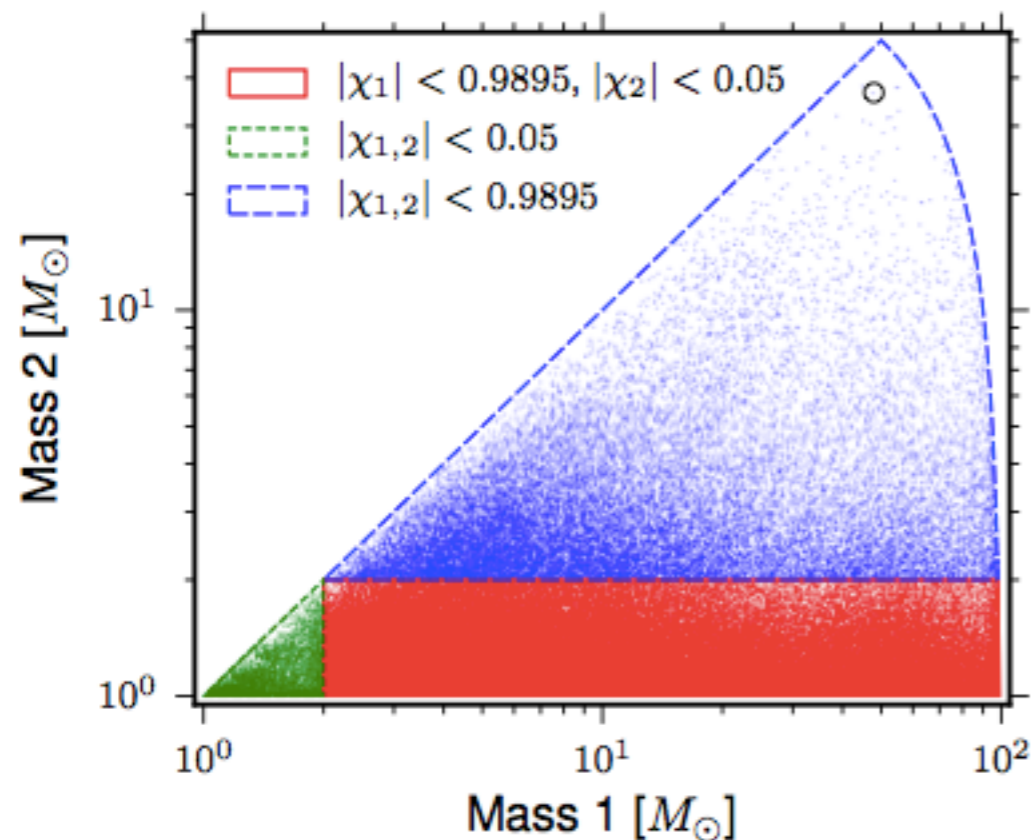


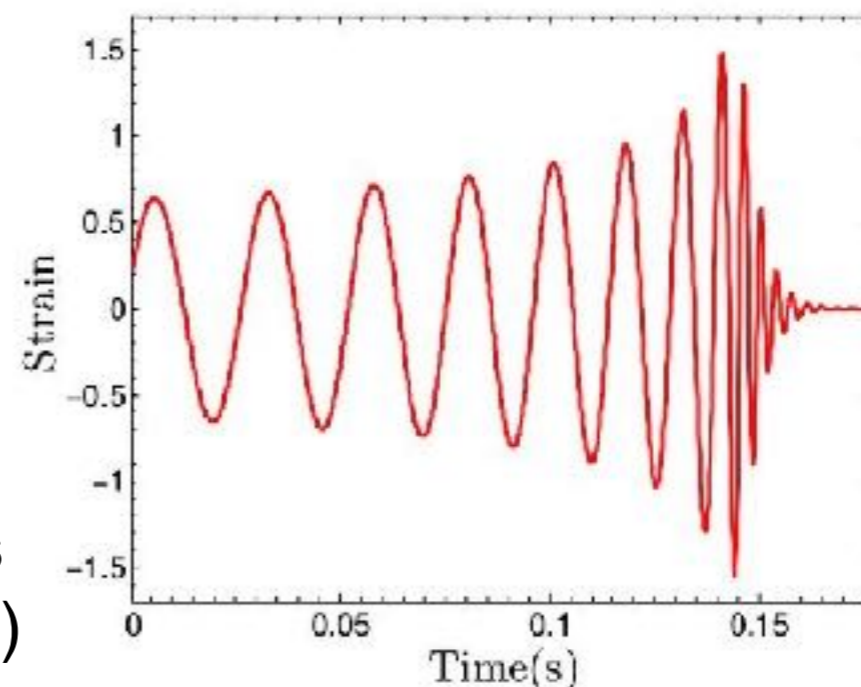
FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2 . Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

Search template bank made of spinning EOB[NR] templates

(Buonanno-Damour99, Damour'01..., Taracchini et al. 14) in ROM form (Puerrer et al.'14);

Recently improved (Bohé et al '17) by including leading 4PN terms (Bini-Damour '13), spin-dependent terms (Pan-Buonanno et al. '13), and calibrating against 141 NR simulations.

[post-computed NR waveform for GW151226 took three months and 70 000 CPU hours !]



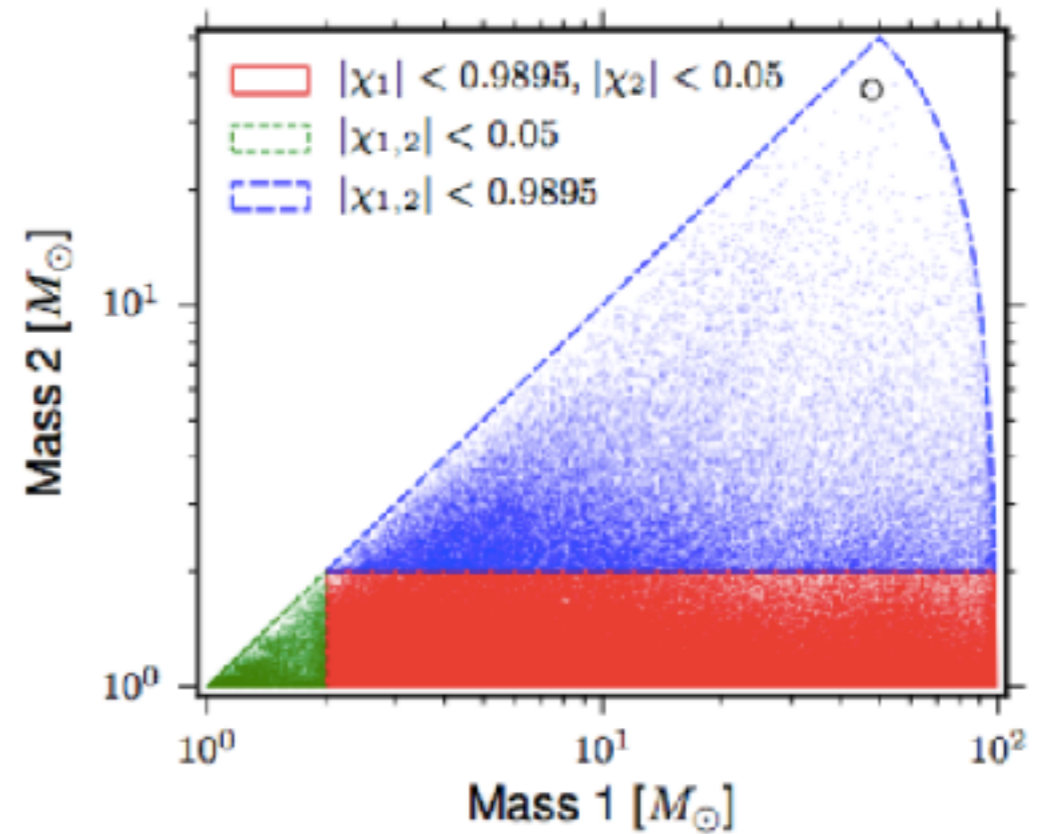
+ auxiliary bank of Phenom[EOB+NR] templates (Ajith...'07, Hannam...'14, Husa...'16, Khan...'16)

LIGO's bank of search templates

O1: 200 000 EOB + 50 000 PN

O2: 325 000 EOB + 75 000 PN

(Taracchini et al.'14, Bohé et al.'17)



PN
 $v \ll c$
 $R \gg GM/c^2$

EOB

LISA's templates
 via EOB[SF] ?

PM
 $R \gg GM/c^2$

NR
 $v \sim c$
 $R \sim GM/c^2$

QFT
 perturbation
 theory

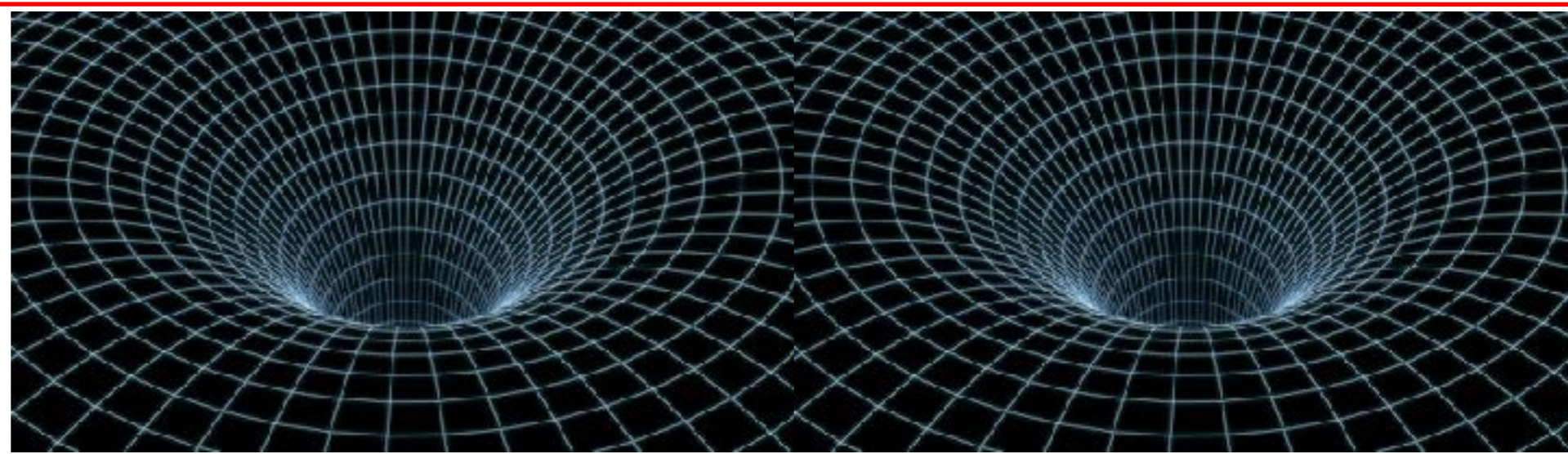
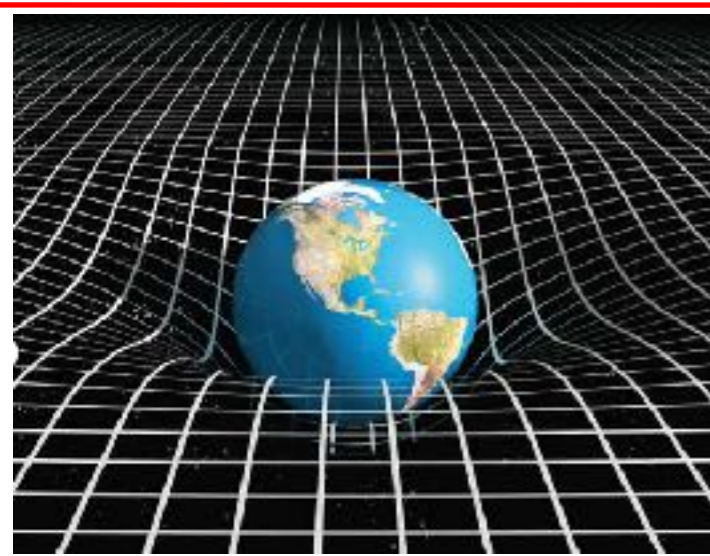
STRING
 perturbation
 theory

SF
 $m_1 \ll m_2$

EOB= Effective-One-Body
 Buonanno-Damour 1999, 2000;
 Damour-Jaranowski-Schaefer 2000;
 Damour 2001

Quantum Scattering Amplitudes

Motion of Strongly Self-gravitating Bodies (NS, BH)

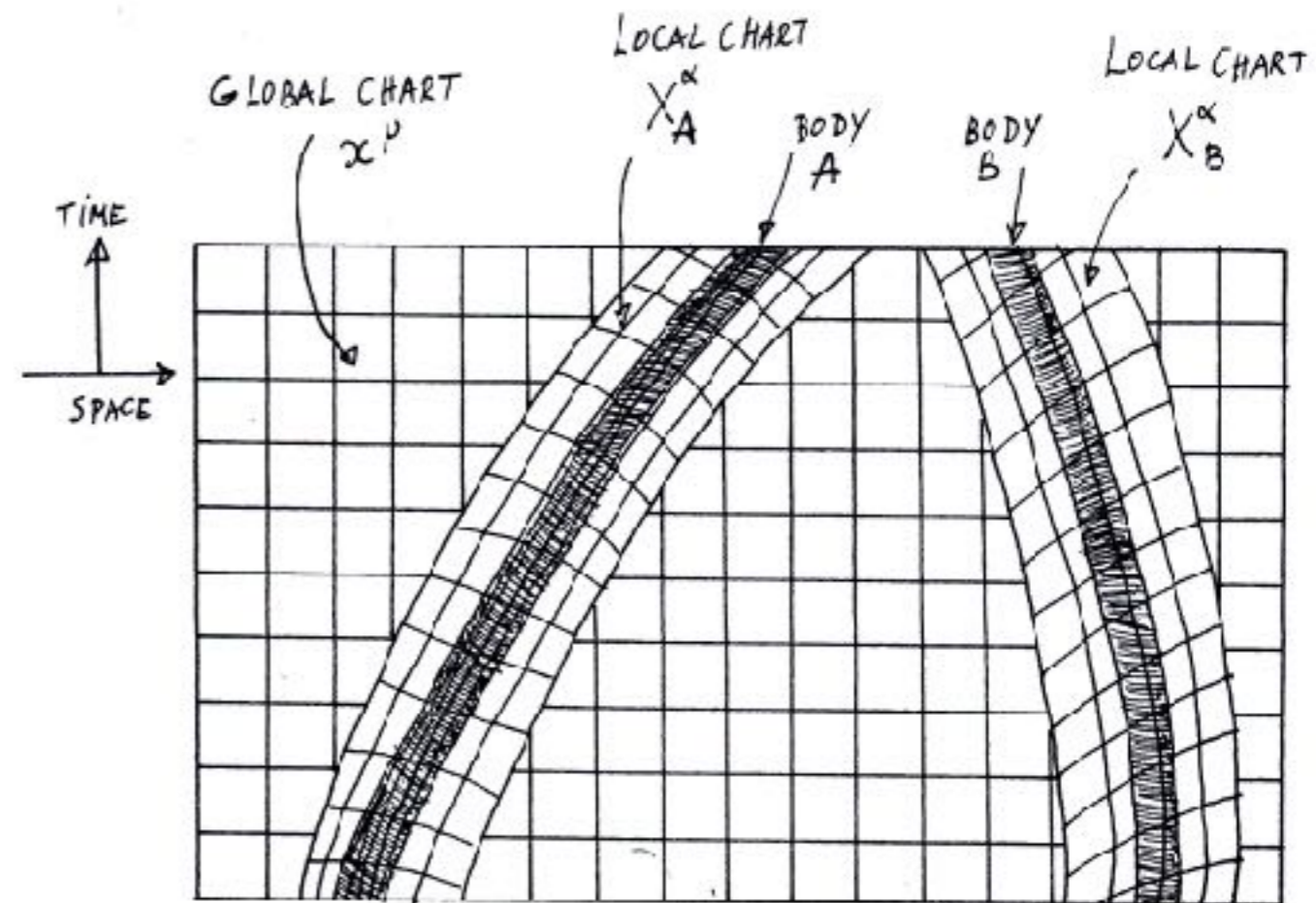


Multi-chart approach to motion of strong-self-gravity bodies, and **matched asymptotic expansions** [EIH '38], Manasse '63, Demianski-Grishchuk '74, D'Eath'75, Kates '80, Damour '82

Useful even for weakly self-gravitating bodies, i.e. "relativistic celestial mechanics", Brumberg-Kopeikin '89, Damour-Soffel-Xu '91-94

Combine two expansions in two charts:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + Gh_{\mu\nu}^{(1)}(x) + G^2 h_{\mu\nu}^{(2)}(x) + \dots \quad G_{\alpha\beta}(x) = G_{\alpha\beta}^{(0)}(x) + G_{\alpha\beta}^{(1)}(x) + \dots$$



Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \rightarrow$ **point-masses** (Mathisson '31)

delta-functions in GR : Infeld '54, Infeld-Plebanski '60

justified by Matched Asymptotic Expansions (« **Effacing Principle** » Damour '83

possible internal-structure dependence in strong self-gravity objects (NSs, BHs)

only arise at 5PN= 5-loop level)

UV divergences linked to self-field effects (loops on external lines) [Dirac, 1938]

QFT's **analytic** (Riesz '49) or **dimensional regularization** (Bollini-Giambiagi '72,

t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer

'01, ...)

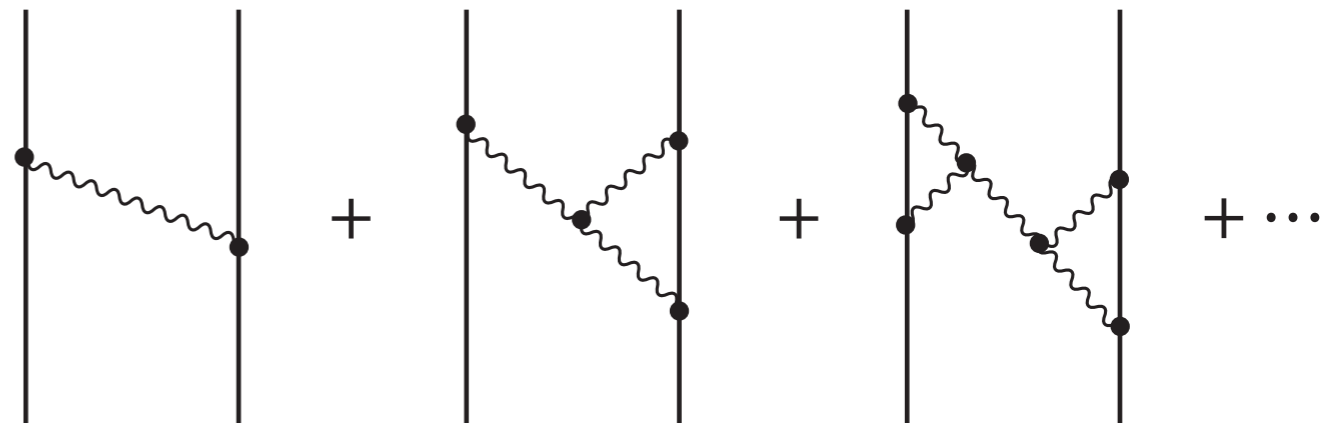
Feynman-like diagrams and
« **Effective Field Theory** » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15



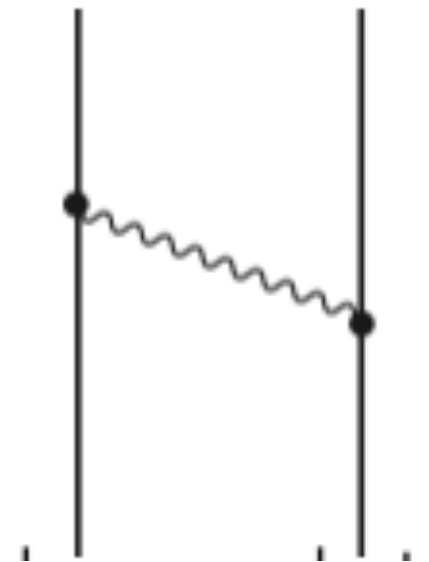
Reduced Worldline Action in Electrodynamics (Fokker 1929)

$$S_{\text{tot}}[x_a^\mu, A_\mu] = - \sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + S_{\text{gf}}$$

« Integrate out » the field A_μ in the total (particle+field) action

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = - \sum_a m_a \int ds_a + \frac{1}{2} \sum_{a,b} e_a e_b \iint dx_a^\mu dx_{b\mu} \delta((x_a - x_b)^2).$$

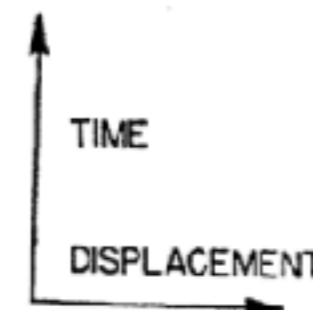
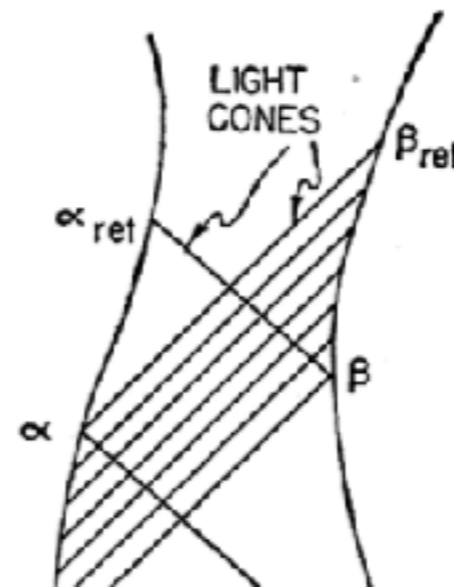
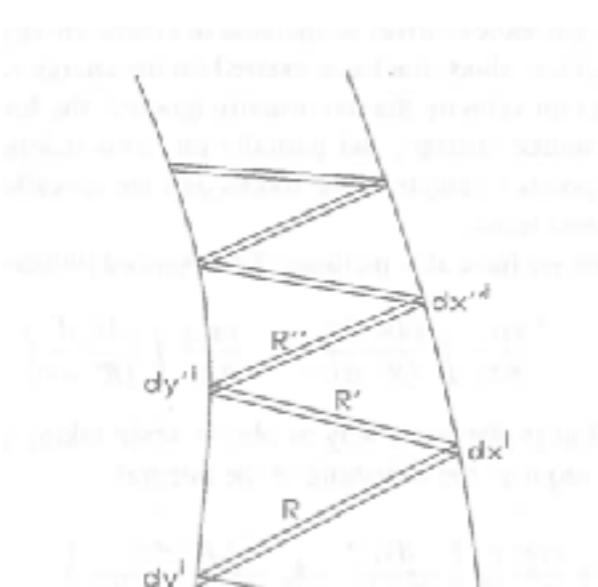
One-photon-exchange diagram



time-symmetric Green function G .

$$G(x) = \delta(-\eta_{\mu\nu} x^\mu x^\nu) = \frac{1}{2r} (\delta(t - r) + \delta(t + r)) ; \square G(x) = -4\pi \delta^4(x)$$

The effective action $S_{\text{eff}}(x_a)$ was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker



Reduced Action in Gravity and its Diagrammatic Expansion

Damour-Esposito-Farese '96

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = [S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}}]_{g_{\mu\nu}(x) \rightarrow g_{\mu\nu}^{\text{gf}}[x_a(s_a)]} :$$

Needs gauge-fixed* action and time-symmetric Green function G .

*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.

Perturbatively solving (in dimension $D=4 + \epsilon$) Einstein's equations to get the equations of motion and the action for the conservative dynamics

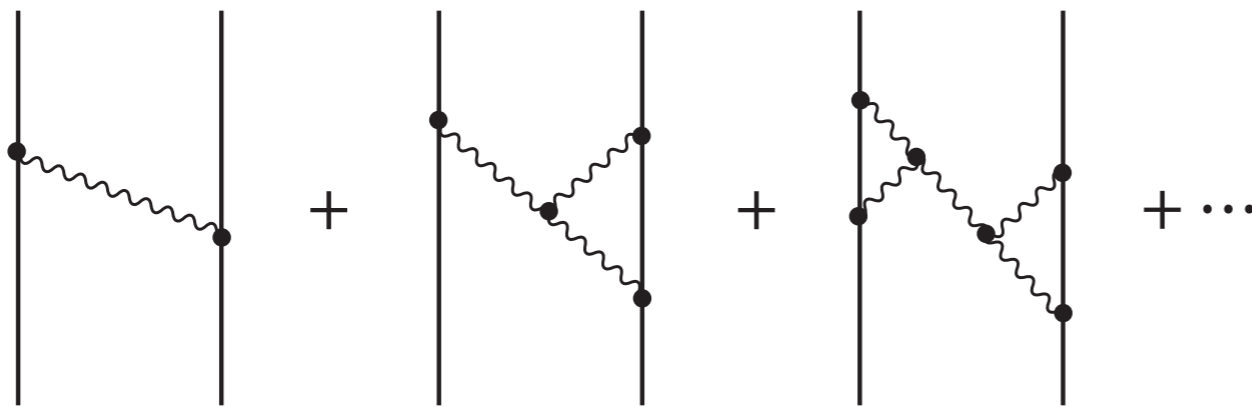
$$g = \eta + h$$

$$S(h, T) = \int \left(\frac{1}{2} h \square h + \partial \partial h h h + \dots + (h + h h + \dots) T \right)$$

$$\square h = -T + \dots \rightarrow h = G T + \dots$$

$$S_{\text{red}}(T) = \frac{1}{2} T G T + V_3(G T, G T, G T) + \dots$$

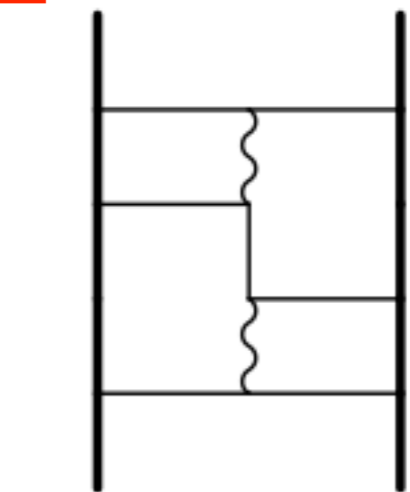
time-symmetric Green function G



$O(G) = 1\text{PM} =$
Newtonian
+ $(v/c)^n$ corrections

$O(G^2) = 2\text{PM}$
= 1 loop \rightarrow 1PN

$O(G^3) = 3\text{PM}$
= 2 loop \rightarrow 2PN



$O(G^5) = 5\text{PM}$
= 4 loop \rightarrow 4PN

Alternative (Equivalent) Computation of Reduced Action

Instead of **on-shell replacing** the field dofs

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = [S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}}]_{g_{\mu\nu}(x) \rightarrow g_{\mu\nu}^{\text{gf}}[x_a(s_a)]}$$

Formal functional integral over the field (QED: Feynman '50; ...;

GR: « **Effective Field Theory** » approach

(Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10, Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15; Foffa-Mastrolia-Sturani-Sturm'16, Damour-Jaranowski '17)

$$e^{\frac{i}{\hbar} S_{\text{eff}}^{\text{quant}}} = \int Dg_{\mu\nu} e^{\frac{i}{\hbar} (S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}})}$$

Saddle-point estimation: $S_{\text{eff}}^{\text{quant}}[x_a(s_a)] = S_{\text{eff}}^{\text{class}}[x_a(s_a)] + O(\hbar)$.

However, the explicit computations are done differently:

by means of **Wick's theorem**, and **p-space integrations**

$$e^{\frac{i}{\hbar} S_{\text{eff}}} = \int D\varphi e^{\frac{i}{\hbar} \int [\frac{1}{2} \varphi \mathcal{K} \varphi + \varphi s + g \varphi^3 + \dots]}$$

$$\int D\varphi e^{\frac{i}{\hbar} \int [\frac{1}{2} \varphi \mathcal{K} \varphi]} \sum_n \frac{(i/\hbar)^n}{n!} \left(\int (\varphi s + g \varphi^3 + \dots) \right)^n$$

$$\langle \varphi(x) \varphi(y) \rangle = \int D\varphi e^{\frac{i}{\hbar} \int [\frac{1}{2} \varphi \mathcal{K} \varphi]} \varphi(x) \varphi(y) = i \hbar \mathcal{K}_{x,y}^{-1}$$

Slow-Motion (PN) computation of the Reduced Gravity Action

Beyond 1-loop order efficient to use **PN-expanded Green function** for explicit computations.

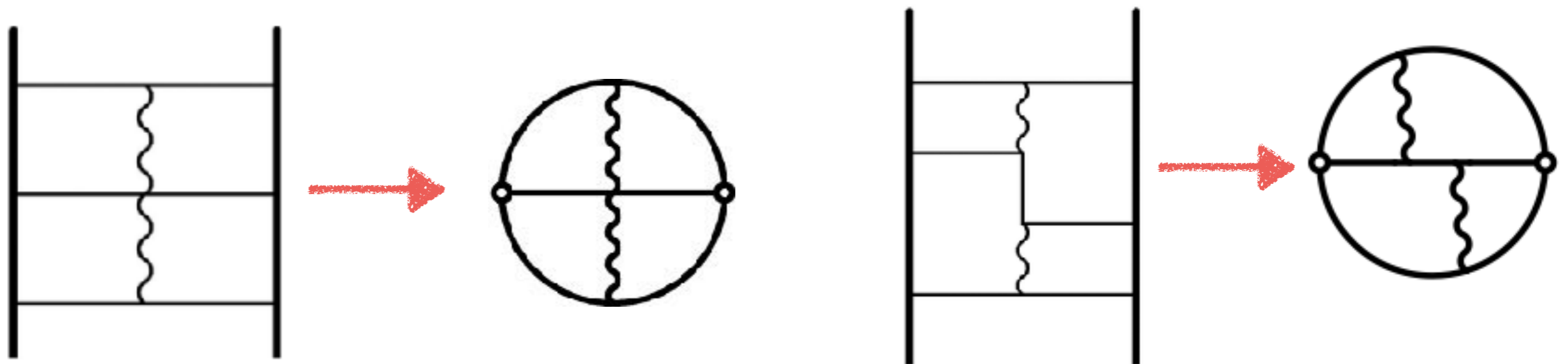
$$\square^{-1} = (\Delta - \frac{1}{c^2} \partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2} \partial_t^2 \Delta^{-2} + \dots$$

PN expansion: in powers of $1/c^2$: 1PN= $(v/c)^2$; 2PN= $(v/c)^4$, etc n PN= $(v/c)^{2n}$

$$G = \delta(-\eta_{\mu\nu} x^\mu x^\nu) \sim \text{PP} \left(\frac{4\pi}{p^2} \right) \rightarrow \frac{\delta(x^0)}{|\mathbf{x}|} + \dots \sim \frac{4\pi \delta(x^0)}{\mathbf{p}^2} + \dots$$

This transforms spacetime diagrams (between two worldlines) into **(massless) two-point space diagrams** (in three dimensions)

E.g. at 4PN, some **4-loop space diagrams** $\sim G^5 m_1^3 m_2^3$ among **515** 4PN-level diagrams (Damour-Jaranowski-Schaefer '14, Bernard et al '16, Foffa et al '17)



Regularization and Renormalization

Like in Dirac 1938, the use of point-particles (delta functions) introduces **UV** divergences (linked to **self-field effects**).

It has been shown (« **Effacing property** » Damour '83) that possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level

Below 5PN (5-loop) point-masses in GR are renormalizable
(Damour '83, Goldberger-Rothstein '06)

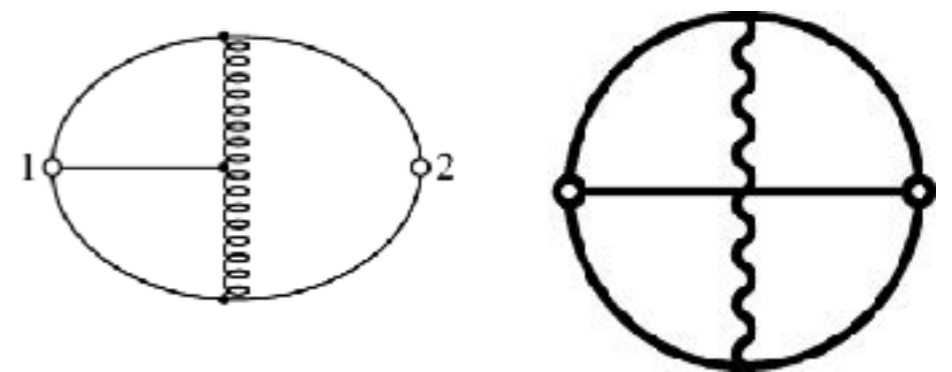
One still needs a regularization method that respects gravity's gauge invariance:
dimensional regularization $D=4+\epsilon$ ('t Hooft-Veltman '72)

It has been explicitly shown that S_{eff} was **UV finite** (in ADM gauge, and in dim.reg.) at **3 loops** (Damour-Jaranowski-Schäfer '01) and

4 loops (Damour-Jaranowski-Schäfer '14, Jaranowski-Schäfer '15)

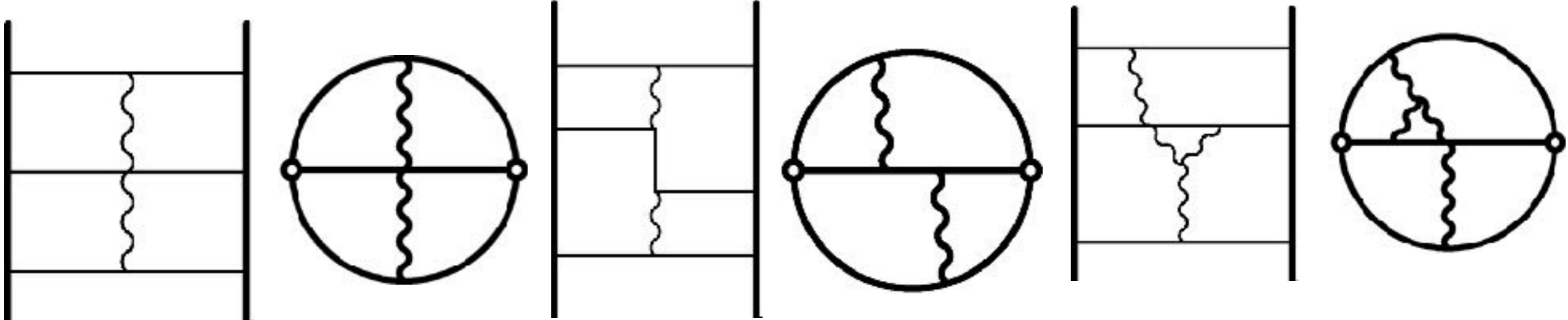
There appear UV ($1/\epsilon$) divergences in harmonic gauge starting at 3PN that can be renormalized away

(Blanchet-Damour-EspositoFarese'04,...,Foffa-Porto-Rothstein-Sturani'19)



There appear **IR divergences** at 4PN (4 loop) linked to non-locality (Blanchet-Damour '88).

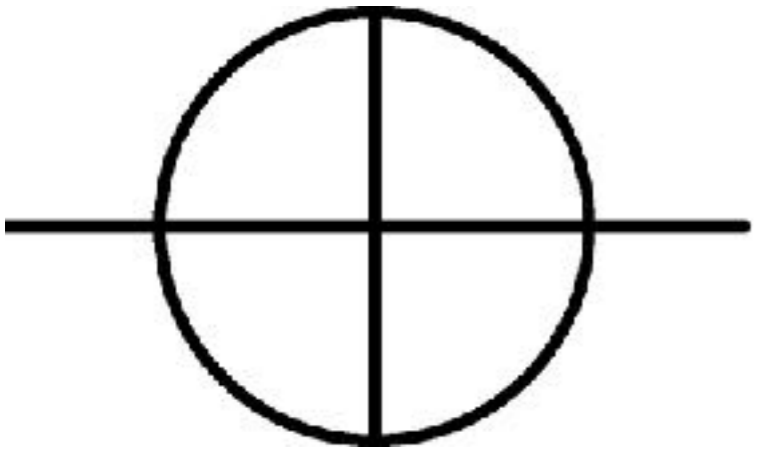
Four-loop static contribution to the gravitational interaction potential of two point masses (Foffa-Mastrolia-Sturani-Sturm '16, Damour-Jaranowski '17)



$$L_{33} = (32 - 2\pi^2) \frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{49} = (64 - 6\pi^2) \frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{50} = \left(-\frac{248}{3} + 8\pi^2 \right) \frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$



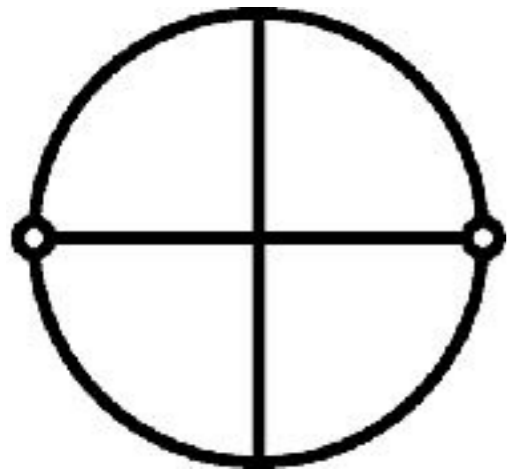
$$\mathcal{M}_{3,6}(\mathbf{p}) = \widehat{\mathcal{M}}_{3,6} |\mathbf{p}|^{4\epsilon-4}, \quad \text{(num.) Lee-Mingulov '15, Damour-Jaranowski '17}$$

where (with γ denoting Euler's constant)

$$\widehat{\mathcal{M}}_{3,6} = (4\pi)^{-4-2\epsilon} \frac{e^{2\gamma\epsilon}}{2} \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{\pi^2}{12} - 8 + O(\epsilon) \right].$$

All integrals computable by the x-space generalized Riesz integral (Jaranowski-Schaefer '00)

$$\int d^3x r_1^a r_2^b (r_1 + r_2 + r_{12})^c = 2\pi R(a, b, c) r_{12}^{a+b+c+3},$$



Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
Blanchet-Damour-Esposito-Farèse '04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
Königsdörffer-Faye-Schäfer '03, Nisanke-Blanchet '05, Itoh '09
- **4PN** (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al.'16-18, Foffa et al.'19)
New feature : **non-locality in time** (Blanchet-Damour'88)
- **Static** two-body potential in harmonic coords at **5PN** (Foffa-Mastrolia-Sturani-Sturm-TorresBobadilla'19, Bluemlein-Maier-Marquard'19)

Inclusion of **spin-dependent effects**: Barker-O'Connell'75, Faye-Blanchet-Buonanno'06,
Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schäfer
'10, Steinhoff'11, Levi-Steinhoff'15-18

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \right) \\ + \frac{1}{4} \frac{Gm_1m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\ + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left(m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\ - \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \frac{G^2 (m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$\begin{aligned}
 c^6 H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left(-14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 & - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} \\
 & + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \\
 & + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \\
 & - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \left. \right) + \frac{G^2 m_1 m_2}{r_{12}^2} \left(\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} \right. \\
 & - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} - \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16} \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^3} + \frac{5}{12} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^3} \\
 & - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)) (\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\
 & + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\
 & - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \left. \right) + \frac{G^3 m_1 m_2}{r_{12}^3} \left(-\frac{1}{48} \left(425m_1^2 + \left(473 - \frac{3}{4}\pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} \right. \\
 & + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(20m_1^2 - \left(43 + \frac{3}{4}\pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\
 & + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \left. \right) \\
 & + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\left(\frac{227}{3} - \frac{21}{4}\pi^2 \right) m_1 - m_2 \right) + (1 \leftrightarrow 2).
 \end{aligned}$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$${}^8H_{4PN}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{7(\mathbf{p}_1^2)^5}{256m^7} + \frac{Gm_1m_2}{r_{12}} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{40}(\mathbf{x}_a, \mathbf{p}_a) \\ + \frac{G^2m_1m_2}{r_{12}^2} (m_1^2 H_{44}(\mathbf{x}_a, \mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\ + \frac{G^4m_1m_2}{r_{12}^4} (m_1^2 H_{12}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2 m_2 H_{122}(\mathbf{x}_a, \mathbf{p}_a)) \\ + \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \quad (\text{A3})$$

$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = \frac{45(\mathbf{p}_1^2)^4}{128m_1^4} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^3}{64m_1^3m_2^3} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^3}{64m_1^3m_2^3} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^2m_2^2} \\ + \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{64m_1^2m_2^2} - \frac{2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{256m_1^2m_2^2} \\ + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^2m_2^2} + \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2^2} - \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2^2} \\ + \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^2m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{64m_1^2m_2^2} \\ + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} - \frac{3(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{256m_1^2m_2^2} \\ + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{128m_1^2m_2^2} - \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{256m_1^2m_2^2} + \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{256m_1^2m_2^2} \\ + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{128m_1^2m_2^2} - \frac{7(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{256m_1^2m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{64m_1^2m_2^2} \\ + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4m_1^2m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} \\ + \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4m_1^2m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} \\ + \frac{(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2^2} - \frac{7(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{128m_1^2m_2^2}. \quad (\text{A4a})$$

$$H_{44}(\mathbf{x}_a, \mathbf{p}_a) = \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{160m_1^4} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{p}_1^2)}{157m_1^3} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^2} - \frac{63(\mathbf{p}_1^2)^3}{64m_1} - \frac{545(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1m_2} \\ + \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)}{15m_1m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^2m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2} - \frac{831(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^2m_2} \\ + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2} - \frac{3253(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^3m_2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{480m_1^3m_2} - \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^3m_2} \\ + \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{350m_1^2m_2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^2m_2} + \frac{4345(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^2m_2} \\ + \frac{3461(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^2m_2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^2m_2} - \frac{1939(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^2m_2} + \frac{2081(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^2m_2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{8m_1^2m_2} \\ + \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^2m_2} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{384m_1^2m_2} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^2m_2} \\ + \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^2m_2} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{96m_1^2m_2} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{96m_1^2m_2} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2} \\ + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{384m_1^2m_2} - \frac{18(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{384m_1^2m_2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4m_1^2m_2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^2m_2} \\ + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{2m_1^2m_2} - \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{6m_1^2m_2} - \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{48m_1^2m_2} \\ + \frac{132(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{24m_1^2m_2} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{96m_1^2m_2} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{96m_1^2m_2} - \frac{173(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{48m_1^2m_2} + \frac{13(\mathbf{p}_1^2)^2}{8m_1^2}. \quad (\text{A4b})$$

$$H_{44}(\mathbf{x}_a, \mathbf{p}_a) = \frac{3127(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22953(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^3}{950m_1^3} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^2} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{640m_1m_2} \\ + \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{1920m_1^2m_2} + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^2m_2} - \frac{752964(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^2m_2} \\ - \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{960m_1^3m_2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4300m_1^3m_2} - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{2400m_1^3m_2} \\ + \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^2m_2} - \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1600m_1^2m_2} - \frac{118261(\mathbf{p}_1^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4800m_1^2m_2} - \frac{105(\mathbf{p}_1^2)^2}{32m_1^2}. \quad (\text{A4c})$$

$$H_{442}(\mathbf{x}_a, \mathbf{p}_a) = \left(\frac{2749\pi^2}{8.92} - \frac{211189}{19200} \right) \frac{(\mathbf{p}_1^2)^2}{m_1^2} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)}{m^2} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\ + \left(\frac{10631\pi^2}{8192} - \frac{1915349}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{12723\pi^2}{16384} - \frac{2492417}{57600} \right) \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2m_2^2} \\ + \left(\frac{1411429}{19200} - \frac{10092\pi^2}{512} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1^2)}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2^2} \\ - \left(\frac{30383}{960} + \frac{36405\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384} \right) \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2} \\ + \left(\frac{2469}{60} + \frac{55655\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2} + \left(\frac{4310\pi^2}{16384} - \frac{39111}{6400} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)}{m_1^2m_2} \\ + \left(\frac{56955\pi^2}{16384} - \frac{1645983}{12200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2}. \quad (\text{A4d})$$

$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861(\mathbf{p}_1^2)}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1m_2} + \frac{105(\mathbf{p}_2^2)}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1m_2}. \quad (\text{A4e})$$

$$H_{422}(\mathbf{x}_a, \mathbf{p}_a) = \left(\frac{1237033}{57600} - \frac{199177\pi^2}{49152} \right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + \left(\frac{282351}{19200} - \frac{21837\pi^2}{8152} \right) \frac{\mathbf{p}_2^2}{m_2^2} \\ + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} - \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \\ + \left(\frac{3200179}{57600} - \frac{28621\pi^2}{24576} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}. \quad (\text{A4f})$$

$$H_{43}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^2}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400} \right) m_1^2 m_2 + \left(\frac{4825\pi^2}{6144} - \frac{603427}{7200} \right) m_1^2 m_2^2. \quad (\text{A4g})$$

$$H_{4PN}^{\text{nonloc}}(t) = -\frac{1}{5} \frac{G^2 M}{c^8} I_{ij}^{(3)}(t) \\ \times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v),$$

Effective One Body (EOB) Method

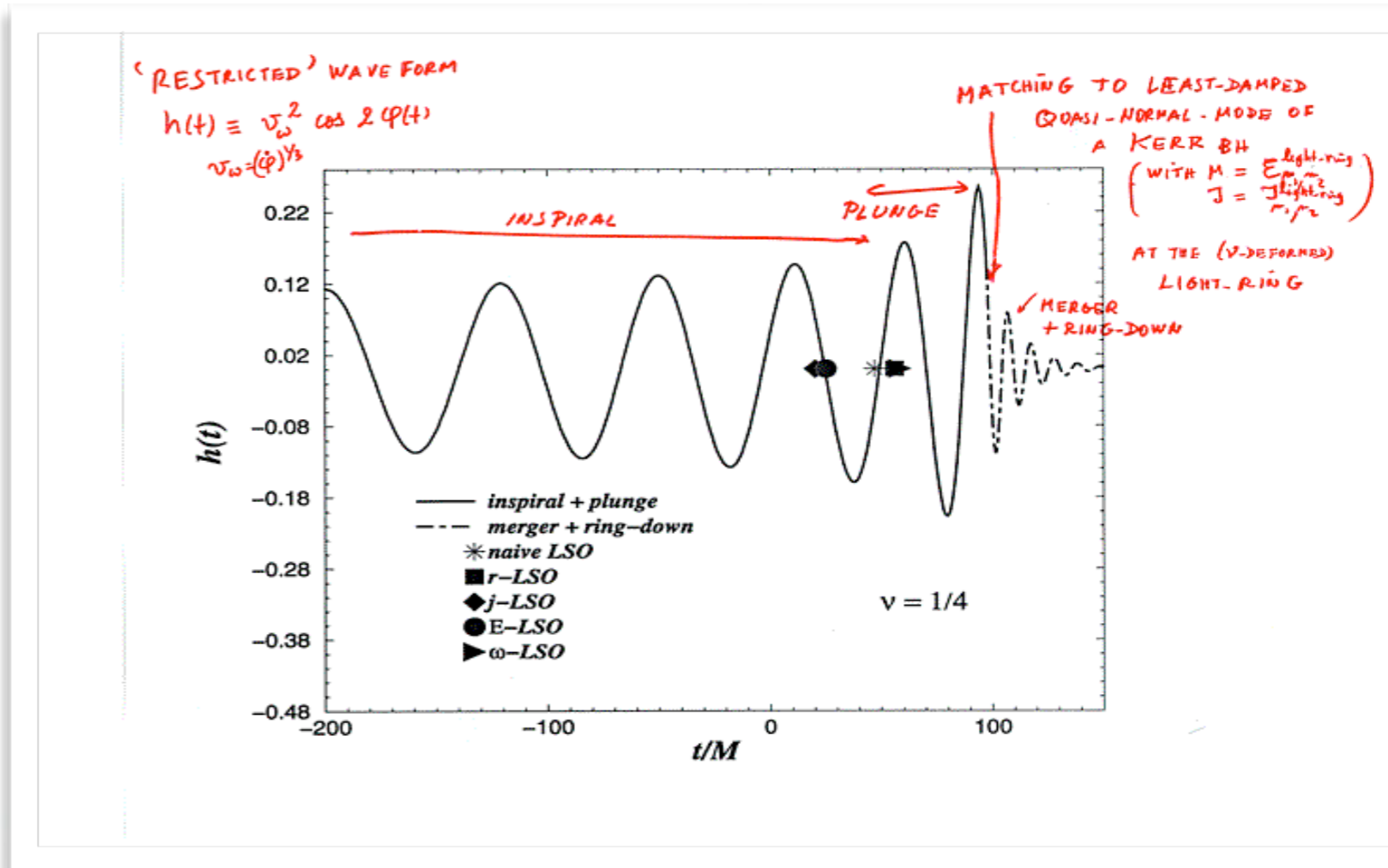
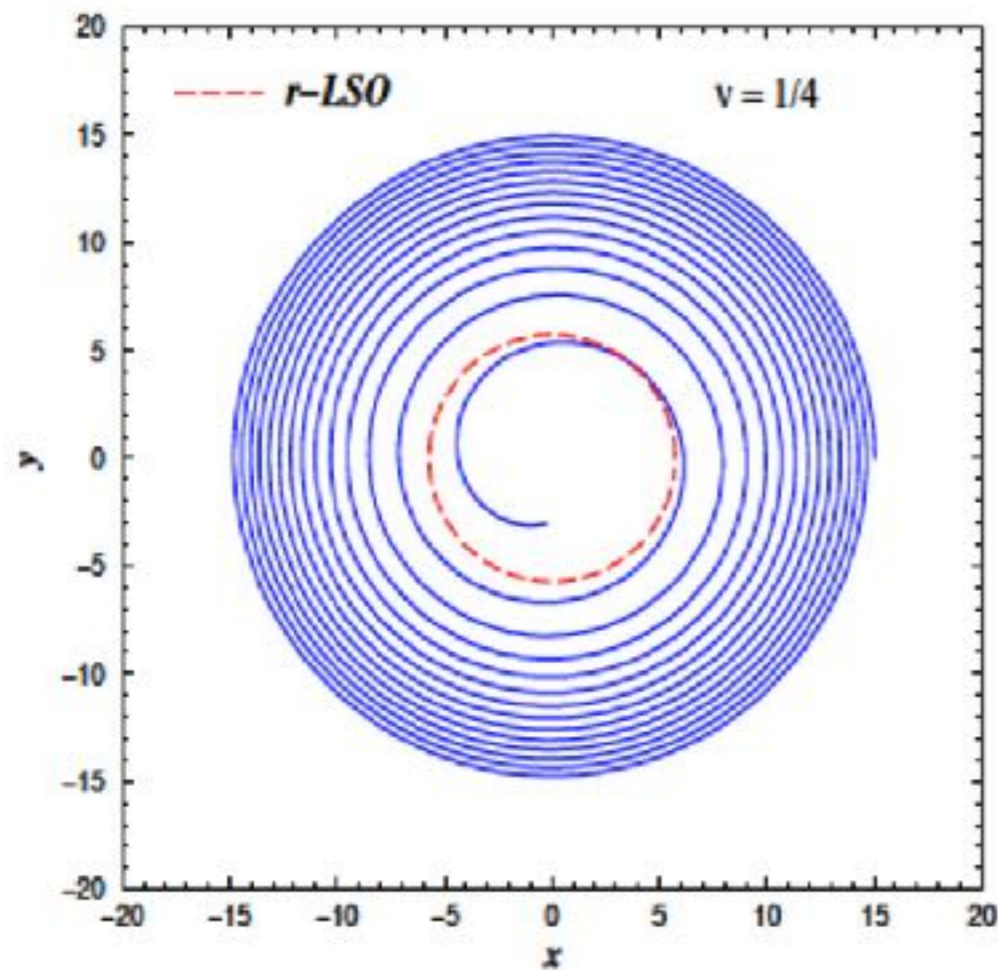
Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001; Damour-Nagar 2007; Damour-Iyer-Nagar 2009

Inability of PN expansions to describe the late inspiral and the merger:

the PN expansions for both the flux and the dynamics reach their radius of convergence

RESUMMATION of both the Hamiltonian, the waveform and radiation-reaction

—> description of the coalescence + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 72)
Buonanno-Damour 2000



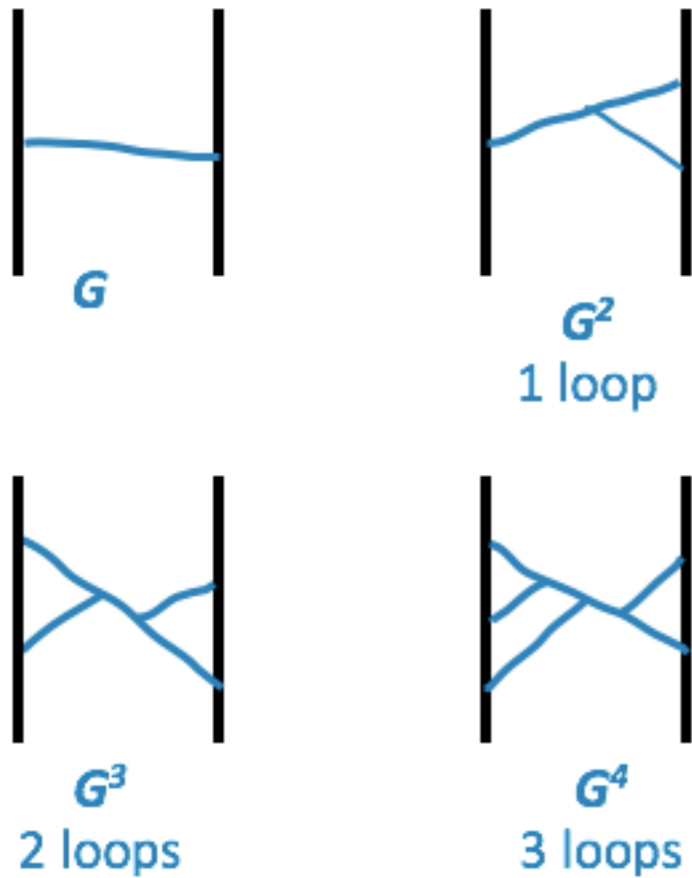
Predictions as early as 2000 :

continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)



An effective particle of mass μ in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\text{eff}}^{\mu\nu}(X)$$



mass-shell constraint

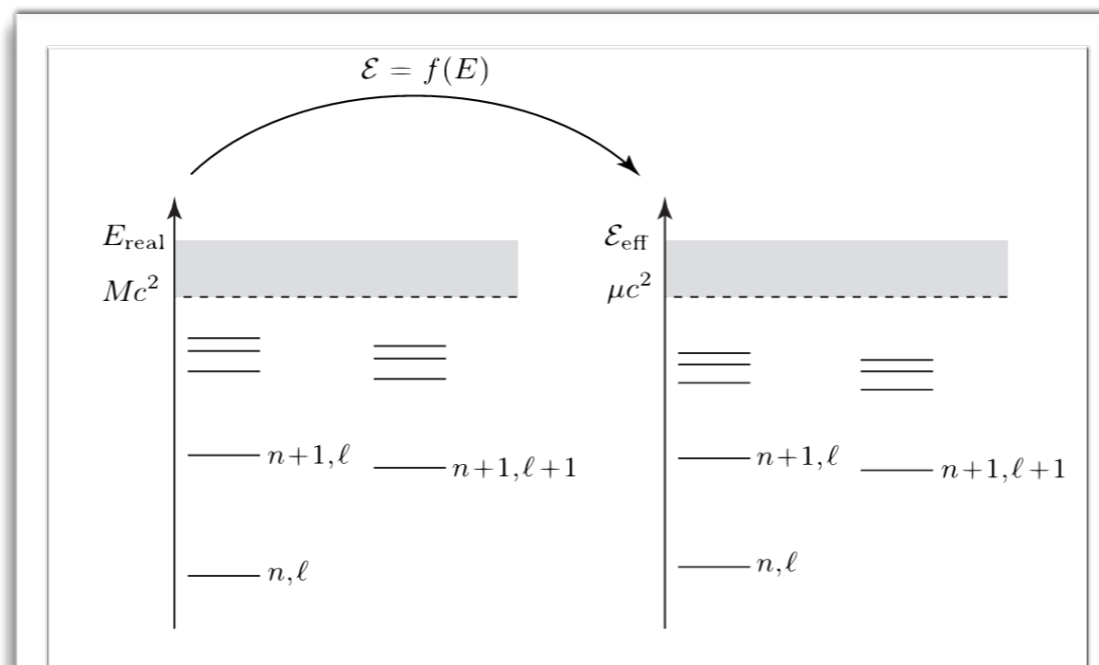
$$0 = g_{\text{eff}}^{\mu\nu}(X) P_\mu P_\nu + \mu^2 + Q(X, P)$$

Level correspondence
in the semi-classical limit:
identification of
quantized action variables

$$J = \ell \hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n \hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$



Crucial energy map

$$\mathcal{E} = f(E)$$

2-body Taylor-expanded N + 1PN + 2PN+ 3PN Hamiltonian

$$\begin{aligned}
 H_N(\mathbf{x}_a, \mathbf{p}_a) &= \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2) \\
 c^2 H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) &= -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \right) \\
 &\quad + \frac{1}{4} \frac{Gm_1m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2), \\
 c^4 H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\
 &\quad \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\
 &\quad + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left(m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\
 &\quad - \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2), \\
 c^6 H_{3PN}(\mathbf{x}_a, \mathbf{p}_a) &= -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1m_2}{r_{12}} \left(-14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 &\quad \left. - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} \right. \\
 &\quad \left. + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \right. \\
 &\quad \left. + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \right. \\
 &\quad \left. - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \right) + \frac{G^2 m_1 m_2}{r_{12}^2} \left(\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} \right. \\
 &\quad \left. - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} - \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371 \mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17 \mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{16 m_1^3} + \frac{5 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{12 m_1^3} \right. \\
 &\quad \left. - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1))(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \right. \\
 &\quad \left. + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\
 &\quad \left. - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) + \frac{G^3 m_1 m_2}{r_{12}^3} \left(-\frac{1}{48} \left(425m_1^2 + \left(473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} \right. \\
 &\quad \left. + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(20m_1^2 - \left(43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \right. \\
 &\quad \left. + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\
 &\quad + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\left(\frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 - m_2 \right) + (1 \leftrightarrow 2).
 \end{aligned}$$

Explicit 3PN EOB dynamics (Damour-Jaranowski-Schaefer '01)

A **simple**, but crucial transformation between the real energy and the effective one:

$$\mathcal{E}_{\text{eff}} = \frac{(\mathcal{E}_{\text{real}})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.$$

A **simple post-geodesic** effective mass-shell:

$$g_{\text{eff}}^{\mu\nu} P'_\mu P'_\nu + \mu^2 c^2 + Q(P'_\mu) = 0,$$

$$ds_{\text{eff}}^2 = -A(R; \nu) dt^2 + B(R; \nu) dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M}$$

$$A^{3\text{PN}}(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41}{32} \pi^2 \right) \nu u^4,$$

$$\bar{D}^{3\text{PN}}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2) u^3,$$

$$\hat{Q}^{3\text{PN}} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2) u^2 \frac{p_r^4}{c^4}.$$

$$u \equiv \frac{GM}{R c^2}$$

Spinning EOB effective Hamiltonian

$$H_{\text{eff}} = H_{\text{orb}} + H_{\text{so}} \quad \rightarrow \quad H_{\text{EOB}} = Mc^2 \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}}{\mu c^2} - 1 \right)}$$

$$\hat{H}_{\text{orb}}^{\text{eff}} = \sqrt{A \left(1 + B_p \mathbf{p}^2 + B_{np} (\mathbf{n} \cdot \mathbf{p})^2 - \frac{1}{1 + \frac{(\mathbf{n} \cdot \boldsymbol{\chi}_0)^2}{r^2}} \frac{(r^2 + 2r + (\mathbf{n} \cdot \boldsymbol{\chi}_0)^2)}{\mathcal{R}^4 + \Delta (\mathbf{n} \cdot \boldsymbol{\chi}_0)^2} ((\mathbf{n} \times \mathbf{p}) \cdot \boldsymbol{\chi}_0)^2 + Q_4 \right)}.$$

$$H_{\text{so}} = G_S \mathbf{L} \cdot \mathbf{S} + G_{S^*} \mathbf{L} \cdot \mathbf{S}^*,$$

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2; \quad \mathbf{S}_* = \frac{m_2}{m_1} \mathbf{S}_1 + \frac{m_1}{m_2} \mathbf{S}_2,$$

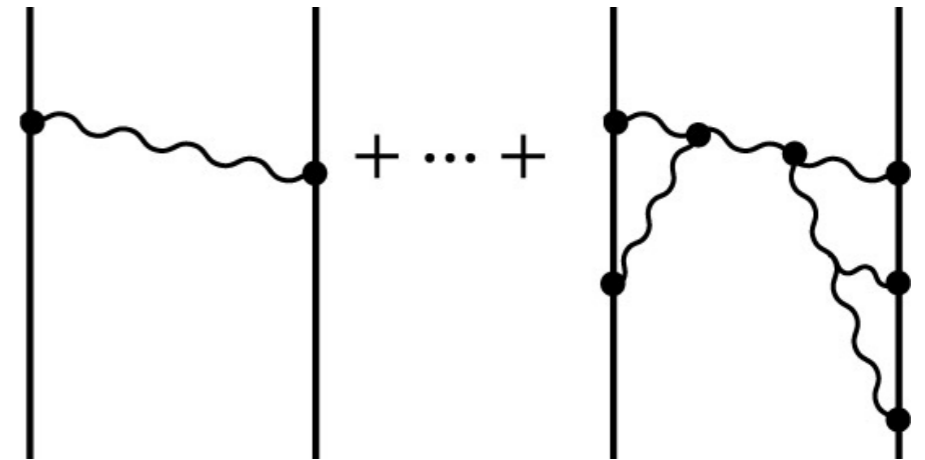
Gyrogravitomagnetic ratios (when neglecting spin² effects)

$$r^3 G_S^{\text{PN}} = 2 - \frac{5}{8} \nu u - \frac{27}{8} \nu p_r^2 + \nu \left(-\frac{51}{4} u^2 - \frac{21}{2} u p_r^2 + \frac{5}{8} p_r^4 \right) + \nu^2 \left(-\frac{1}{8} u^2 + \frac{23}{8} u p_r^2 + \frac{35}{8} p_r^4 \right)$$

$$r^3 G_{S^*}^{\text{PN}} = \frac{3}{2} - \frac{9}{8} u - \frac{15}{8} p_r^2 + \nu \left(-\frac{3}{4} u - \frac{9}{4} p_r^2 \right) - \frac{27}{16} u^2 + \frac{69}{16} u p_r^2 + \frac{35}{16} p_r^4 + \nu \left(-\frac{39}{4} u^2 - \frac{9}{4} u p_r^2 + \frac{5}{2} p_r^4 \right) + \nu^2 \left(-\frac{3}{16} u^2 + \frac{57}{16} u p_r^2 + \frac{45}{16} p_r^4 \right)$$

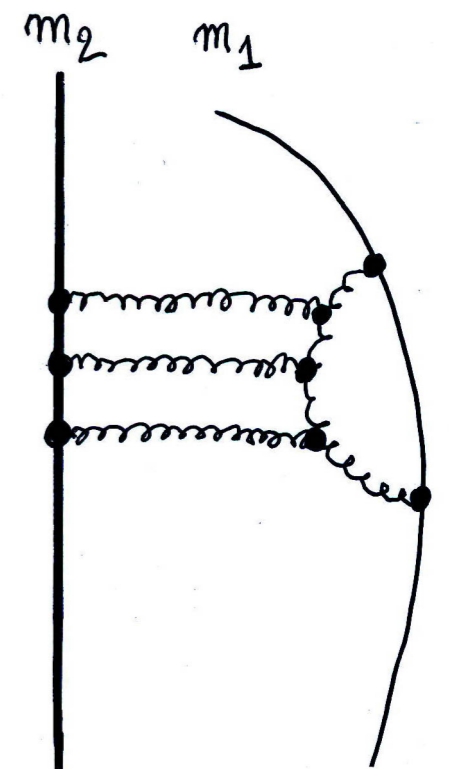
EOB AND GSF

Comparable-mass case: $m_1 \sim m_2$



Gravitational Self-Force Theory : $m_1 \ll m_2$
based on BH perturbation theory:
Regge-Wheeler-Zerilli-Teukolsky

- Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting '10, Damour '10, Blanchet et al '10, LeTiec et al '12, Bini-Damour '13-15, Kavanagh-Ottewill-Wardell '15 Bini-Damour-Geralico'16, Hopper-Kavanagh-Ottewill'16
- (gauge-invariant) Numerical results : Detweiler '08, Barack-Sago '09, Blanchet-Detweiler-LeTiec-Whiting '10, Barack-Damour-Sago '10, Shah-Friedman-Keidl '12, Dolan et al '14, Nolan et al '15, Akcay-van de Meent '16
- Analytical PN results from high-precision (**hundreds to thousands** of digits !) numerical results : Shah-Friedman-Whiting '14, Johnson-McDaniel-Shah-Whiting '15



GSF : ANALYTICAL HIGH-ORDER_PN RESULTS (22 LOOPS)

Bini-Damour 15

$$A(u; \nu) = 1 - 2u + \nu a(u) + O(\nu^2)$$

Kavanagh et al 15

$$a_{10}^c = \frac{18605478842060273}{7079830758000} \ln(2) - \frac{1619008}{405} \zeta(3) - \frac{21339873214728097}{1011404394000} \gamma$$

$$+ \frac{27101981341}{100663296} \pi^6 - \frac{6236861670873}{125565440} \ln(3) + \frac{360126}{49} \ln(2) \ln(3) + \frac{180063}{49} \ln(3)^2$$

$$- \frac{121494974752}{9823275} \ln(2)^2 - \frac{24229836023352153}{549755813888} \pi^4 + \frac{1115369140625}{124540416} \ln(5) + \frac{96889}{2779}$$

$$+ \frac{75437014370623318623299}{18690753201120000} - \frac{60648244288}{9823275} \ln(2) \gamma + \frac{200706848}{280665} \gamma^2$$

$$+ \frac{11980569677139}{2306867200} \pi^2 + \frac{360126}{49} \gamma \ln(3),$$

$$a_{10}^{\ln} = -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665} \gamma - \frac{30324122144}{9823275} \ln(2) + \frac{180063}{49} \ln(3),$$

$$a_{10}^{\ln^2} = \frac{50176712}{280665},$$

$$a_{10.5}^c = -\frac{185665618769828101}{24473489040000} \pi + \frac{377443508}{77175} \ln(2) \pi + \frac{2414166668}{1157625} \pi \gamma - \frac{5846788}{11025} \pi^3 - \frac{2}{2}$$

$$a_{10.5}^{\ln} = \frac{1207083334}{1157625} \pi.$$

**Numerical GSF computation
of the $O(\nu)$ $A(u; \nu)$ -potential:
with singularity at $u=1/3$
(Akçay et al 2012)**

$$a(u) \simeq 0.25(1 - 3u)^{-1/2}$$

**GSF can also bring
scattering information**

(Damour 2010, Barack et al, in prepar)

$$c_{15} = -\frac{2069543450583769619340376724}{325477442086506084375} \zeta(3) + \frac{65195026298245007936}{22370298575625} \gamma \zeta(3) - \frac{5049442304}{25725} \gamma^2 \zeta(3) + \frac{1262360576}{15435} \pi^2 \zeta(3)$$

$$+ \frac{171722752}{441} \zeta(3)^2 + \frac{1613866959570176}{496621125} \zeta(5) - \frac{343445504}{441} \gamma \zeta(5) - \frac{146997248}{105} \zeta(7) + \frac{56314978304}{385875} \zeta(3) \log^2(2)$$

$$- \frac{106445664}{343} \zeta(3) \log^2(3) + \frac{151670998244849797696}{22370298575625} \zeta(3) \log(2) - \frac{190336581632}{1157625} \gamma \zeta(3) \log(2)$$

$$+ \frac{28863591064624341}{4909804900} \zeta(3) \log(3) - \frac{212891328}{343} \gamma \zeta(3) \log(3) - \frac{212891328}{343} \zeta(3) \log(2) \log(3) - \frac{77186767578125}{19876428} \zeta(3) \log$$

$$- \frac{2039263232}{3675} \zeta(5) \log(2) - \frac{49128768}{49} \zeta(5) \log(3) + \frac{298267427515018397019736592175289419501391539444290849}{6587612222544653226142468405031917319531250}$$

$$- \frac{6807661768453637768313286948060329087501419}{704310948124803722562607729544062500} \gamma + \frac{1598346944412603247831006289829388}{526171715038677033591890625} \gamma^2 - \frac{1007647146215971027644}{335890033113009375}$$

$$+ \frac{461219496448}{72930375} \gamma^4 - \frac{28338275082077591587855063450276303790065762907243197}{999703155845143418115744045792755712000000} \pi^2 + \frac{25191178655399275691104}{67178006622601875} \gamma \pi^2$$

$$- \frac{230609748224}{14586075} \gamma^2 \pi^2 + \frac{10548032335775226894713787760391180776248036241}{304245354831316028025099055320268800000} \pi^4 + \frac{1262360576}{385875} \gamma \pi^4$$

$$- \frac{6208472839612966972691457131143}{266930151354100246118400} \pi^6 + \frac{3573178781920929118281329}{151996487423754240} \pi^8 - \frac{10136323685888}{72930375} \log^4(2) + \frac{38438712}{2401} \log^4(3)$$

$$- \frac{177896086126482679647872}{54963823600310625} \log^3(2) - \frac{89686013106176}{364651875} \gamma \log^3(2) + \frac{153754848}{2401} \log^3(2) \log(3)$$

$$- \frac{131463845322790269123}{245735735245000} \log^3(3) + \frac{153754848}{2401} \gamma \log^3(3) + \frac{153754848}{2401} \log(2) \log^3(3) + \frac{11933074267578125}{51161925672} \log^3(5)$$

$$+ \frac{3878258674166628974595420635200204}{189421817413923732093080625} \log^2(2) - \frac{3440856379914601692151168}{1007670099339028125} \gamma \log^2(2) - \frac{16582891400192}{121550625} \gamma^2 \log^2(2)$$

$$+ \frac{4145722850048}{72930375} \pi^2 \log^2(2) - \frac{523697163373483905609}{245735735245000} \log^2(2) \log(3) + \frac{461264544}{2401} \gamma \log^2(2) \log(3)$$

$$+ \frac{45454535766189065888302299261759}{6569728226789883034880000} \log^2(3) - \frac{394391535968370807369}{245735735245000} \gamma \log^2(3) + \frac{230632272}{2401} \gamma^2 \log^2(3)$$

$$- \frac{96096780}{2401} \pi^2 \log^2(3) - \frac{437493411770075173449}{245735735245000} \log(2) \log^2(3) + \frac{461264544}{2401} \gamma \log(2) \log^2(3)$$

$$+ \frac{230632272}{2401} \log^2(2) \log^2(3) + \frac{11933074267578125}{17053975224} \log^2(2) \log(5) - \frac{2505842696993145943705498046875}{402136320895332222431232} \log^2(5)$$

$$+ \frac{11933074267578125}{17053975224} \gamma \log^2(5) + \frac{11933074267578125}{17053975224} \log(2) \log^2(5) + \frac{47929508316470415142010251}{56464635170211840000} \log^2(7)$$

$$- \frac{181636067216895220421537747685253699734494659}{6338798533123233503063469565896562500} \log(2) + \frac{74203662155219108543799531653010136}{4735545435348093302327015625} \gamma \log(2)$$

$$- \frac{1482169326522492515499392}{1007670099339028125} \gamma^2 \log(2) - \frac{4905667647488}{364651875} \gamma^3 \log(2) + \frac{371228115490667668451168}{604602059603416875} \pi^2 \log(2)$$

$$+ \frac{1226416911872}{72930375} \gamma \pi^2 \log(2) + \frac{23792072704}{17364375} \pi^4 \log(2) - \frac{4141158375397180302387095124935855747727}{108266631596274488880198656000000} \log(3)$$

$$+ \frac{9459358001131575454332055276239}{691550339662092951040000} \gamma \log(3) - \frac{394391535968370807369}{245735735245000} \gamma^2 \log(3) + \frac{153754848}{2401} \gamma^3 \log(3)$$

$$+ \frac{131463845322790269123}{196588588196000} \pi^2 \log(3) - \frac{192193560}{2401} \gamma \pi^2 \log(3) + \frac{8870472}{1715} \pi^4 \log(3)$$

$$+ \frac{214411501060211389845962927148381}{13139456453579766069760000} \log(2) \log(3) - \frac{437493411770075173449}{122867867622500} \gamma \log(2) \log(3)$$

$$+ \frac{461264544}{2401} \gamma^2 \log(2) \log(3) - \frac{192193560}{2401} \pi^2 \log(2) \log(3) + \frac{978612948501709853277095576118865234375}{17942749191956127021132384903168} \log(5)$$

$$- \frac{2505842696993145943705498046875}{201068160447666111215616} \gamma \log(5) + \frac{11933074267578125}{17053975224} \gamma^2 \log(5) - \frac{59665371337890625}{204647702688} \pi^2 \log(5)$$

$$- \frac{2505842696993145943705498046875}{201068160447666111215616} \log(2) \log(5) + \frac{11933074267578125}{8526987612} \gamma \log(2) \log(5)$$

$$- \frac{5858006173792308915665113013914648081}{32391919320751280297792000000} \log(7) + \frac{47929508316470415142010251}{28232317585105920000} \gamma \log(7)$$

$$+ \frac{47929508316470415142010251}{28232317585105920000} \log(2) \log(7) + \frac{7400249944258160101211}{65676344832000000} \log(11),$$

Nonlocal-in-time (IR divergent) 4PM and 4PN contribution

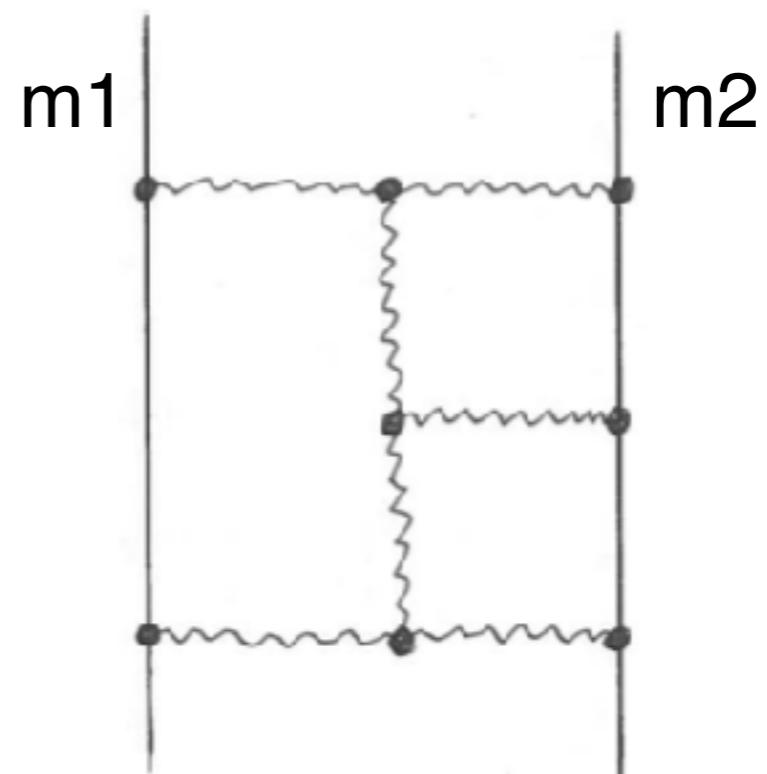
Blanchet-Damour'88: nonlocality in dynamics arises at 4PN via tail term
 Damour-Jaranowski-Schaefer'14: Nonlocal contribution to 4PN reduced action

$$H_{4\text{PN}}^{\text{nonloc}}(t) = -\frac{1}{5} \frac{G^2 M}{c^8} I_{ij}^{(3)}(t) \times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v), \quad + C \frac{2}{5} \frac{G^2 M}{c^8} (I_{ij}^{(3)})^2.$$

Constant C obtained by matching the PN-ambiguous near-zone result to a well-defined computation **using a global Green function valid both in near-zone and wave-zone** (Bini-Damour'13)

$$C = -\frac{1681}{1536}$$

4PM diagrams leading to problematic 4PN ones are $O(G^4 m_1^2 m_2^3) = 1\text{SF}$

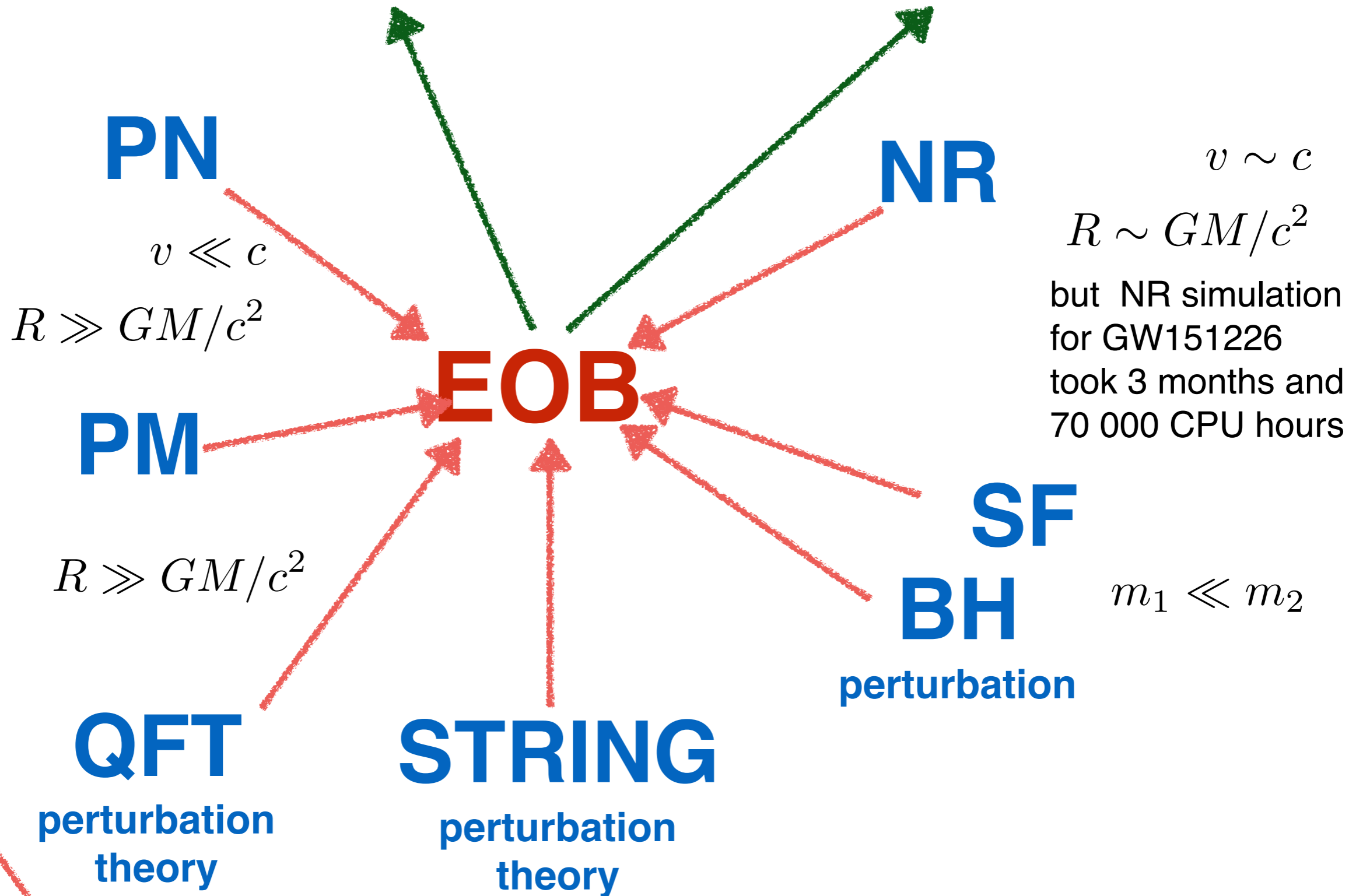


Bernard et al.'18 and Foffa et al.'19 claim to recover our result without explicit near-zone-wave-zone matching¹⁵

**G
R
2-
B
O
D
Y
P
R
O
B
L
E
M**

LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN

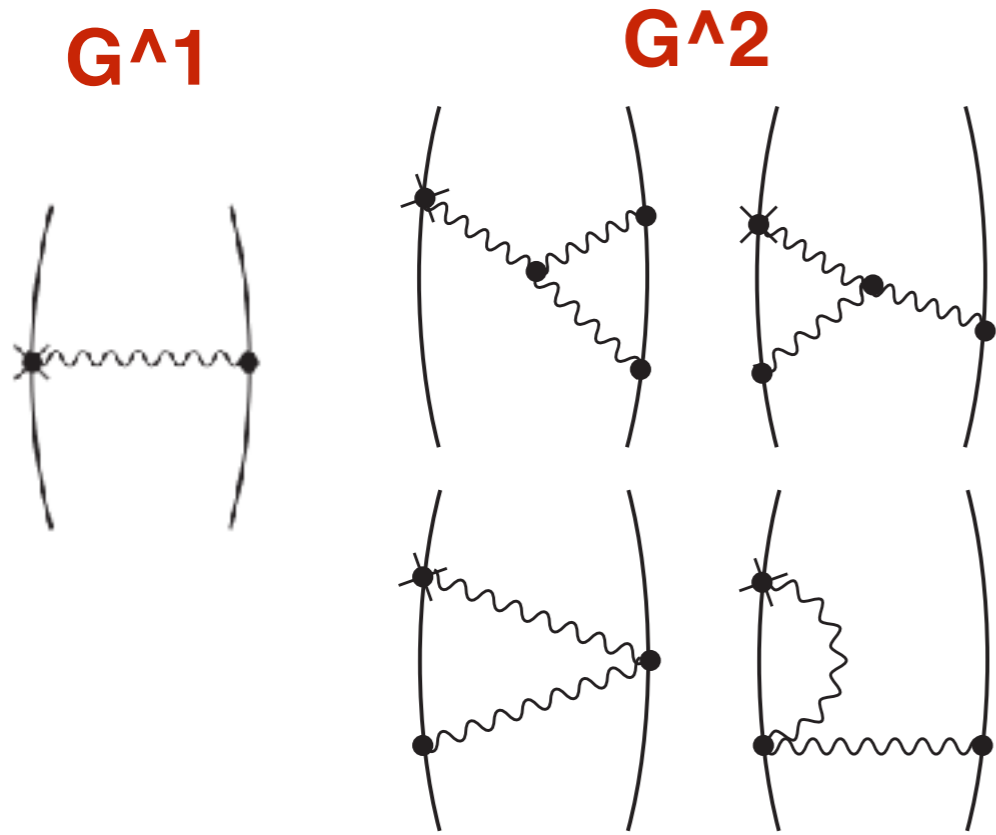
LISA's templates
via EOB[SF] ?



Quantum Scattering Amplitudes

PM Perturbation Theory for Classical Gravitational Scattering

Damour '18, using Bel-Martin '75-'81, Portilla '79, Westpfahl-Goller '79, Portilla '80, Bel-Damour-Deruelle-Ibanez-Martin '81, Westpfahl '85



+
ladders

$$\Delta p_{1\mu} = 2G \int d\sigma_1 d\sigma_2 p_{1\alpha} p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta; \alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2)) p_{2\alpha'} p_{2\beta'} + 2G \int d\sigma_1 d\sigma_1' p_{1\alpha} p_{1\beta} \partial_\mu \mathcal{P}^{\alpha\beta; \alpha'\beta'}(x_1(\sigma_1) - x_1(\sigma_1')) p_{1\alpha'} p_{1\beta'} + O(G^2)$$

$$\Delta p_{1\mu} = 8\pi G \int \frac{d^4 k}{(2\pi)^4} i k_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta; \alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'} \times \int d\sigma_1 \int d\sigma_2 e^{ik \cdot (x_1(\sigma_1) - x_2(\sigma_2))}$$

$$\frac{1}{2} \chi_{\text{class}}(E, J) = \frac{1}{j} \chi_1(\hat{E}_{\text{eff}}, \nu) + \frac{1}{j^2} \chi_2(\hat{E}_{\text{eff}}, \nu) + O(G^3)$$

$$j \equiv \frac{J}{G m_1 m_2}$$

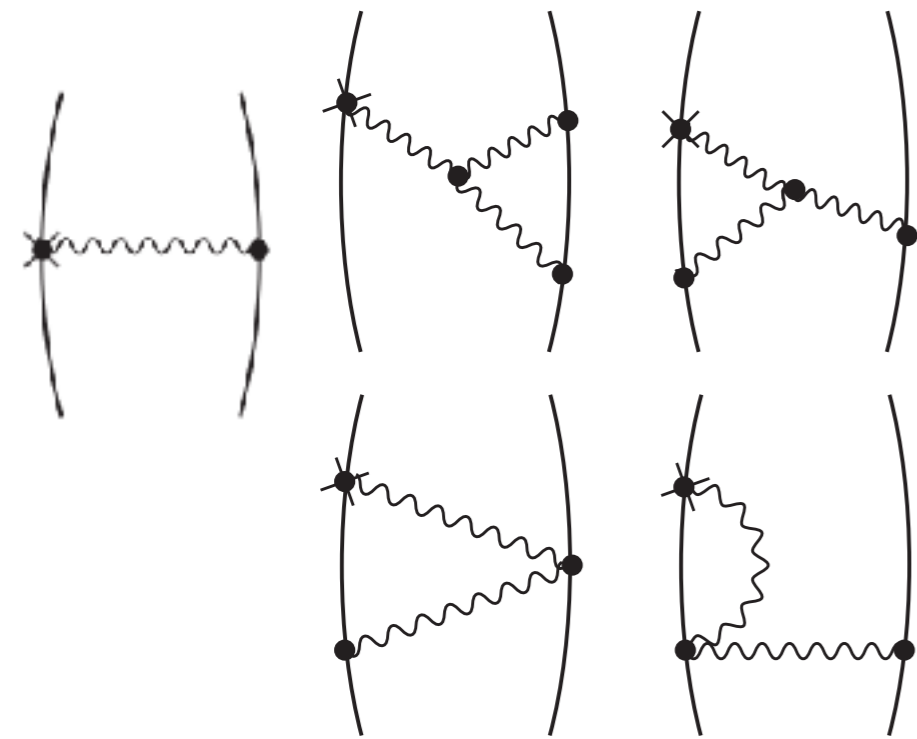
$$\frac{1}{2} \chi_{1PM}^{\text{real}} = \frac{G}{J} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}}$$

$$\hat{\mathcal{E}}_{\text{eff}} \equiv \frac{\mathcal{E}_{\text{eff}}}{\mu} \equiv \frac{(E_{\text{real}})^2 - m_1^2 - m_2^2}{2m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}$$

$$\chi_1(\hat{\mathcal{E}}_{\text{eff}}, \nu) = \frac{2\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\text{eff}}^2 - 1}}$$

$$\chi_2(\hat{\mathcal{E}}_{\text{eff}}, \nu) = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\text{eff}} - 1)}}$$

Classical Gravitational Scattering and the EOB description of the GR 2-body problem



$$0 = g_{\text{eff}}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

Using

$$\frac{1}{2}\chi = \Phi(E_{\text{real}}, J; m_1, m_2, G)$$

At each order in G : EOB transcription of the gauge-invariant scattering function

as a simple energy-dependent modification of a Schwarzschild-metric mass-shell condition

$g_{\text{eff}}^{\mu\nu}$

Schwarzschild metric $M=m_1+m_2$

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

2PM

3PM

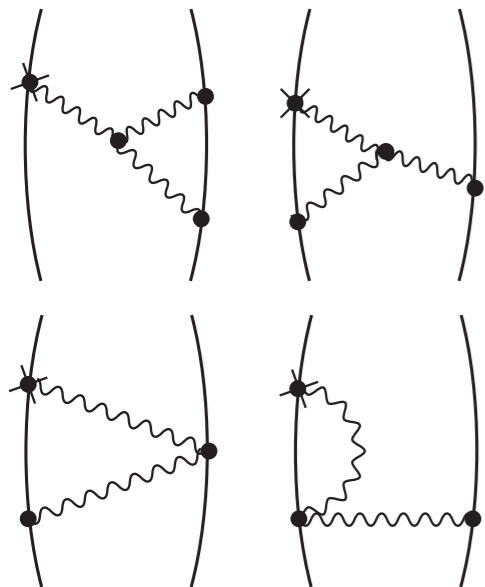
Simple 2PM (one-loop) EOB Potential

G¹

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-IbanezMartin'81, Westpfahl '85



G²



$$0 = g_{\text{Schw}}^{\mu\nu} P_\mu P_\nu + \mu^2 + q_2(E) (GM/R)^2$$

**Schwarzschild
metric $M=m_1+m_2$**

$$\mathcal{E}_{\text{eff}} = \frac{(\mathcal{E}_{\text{real}})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}$$

$$\chi_2(\hat{\mathcal{E}}_{\text{eff}}, \nu) = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\text{eff}} - 1)}}$$

$$q_2(\hat{H}_{\text{Schw}}, \nu) = \frac{3}{2} (5\hat{H}_{\text{Schw}}^2 - 1) \left[1 - \frac{1}{\sqrt{1 + 2\nu(\hat{H}_{\text{Schw}} - 1)}} \right]$$

General Properties of $Q(P;\nu)$ Potential (Damour'18,19)

Mass-ratio dependence: $\nu = m_1 m_2 / (m_1 + m_2)^2$

$$h(\gamma; \nu) = \sqrt{s} / (m_1 + m_2) = \sqrt{1 + 2\nu(\gamma - 1)}$$

$$\begin{aligned} q_2(\gamma, \nu) &= \hat{q}_2^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right) \\ q_3(\gamma, \nu) &= \hat{q}_3^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right) + \hat{q}_3^{(2)}(\gamma) \left(1 - \frac{1}{h^2(\gamma, \nu)} \right) \\ q_4(\gamma, \nu) &= \hat{q}_4^{(1)}(\gamma) \left(1 - \frac{1}{h(\gamma, \nu)} \right) + \hat{q}_4^{(2)}(\gamma) \left(1 - \frac{1}{h^2(\gamma, \nu)} \right) \\ &\quad + \hat{q}_4^{(3)}(\gamma) \left(1 - \frac{1}{h^3(\gamma, \nu)} \right) \end{aligned} \quad (9.10)$$

Some functions already known

$$\hat{q}_2^{(1)}(\gamma) = \frac{3}{2}(5\gamma^2 - 1) \quad \hat{q}_3^{(1)}(\gamma) = -\frac{2\gamma^2 - 1}{\gamma^2 - 1} q_2^{(1)}(\gamma) = -\frac{3}{2} \frac{(2\gamma^2 - 1)(5\gamma^2 - 1)}{\gamma^2 - 1}$$

High-energy Behavior

$$\hat{q}_n^{(p)}(\gamma) \stackrel{\text{HE}}{\sim} \gamma^2$$

1PM and 2PM-accurate spin-orbit couplings

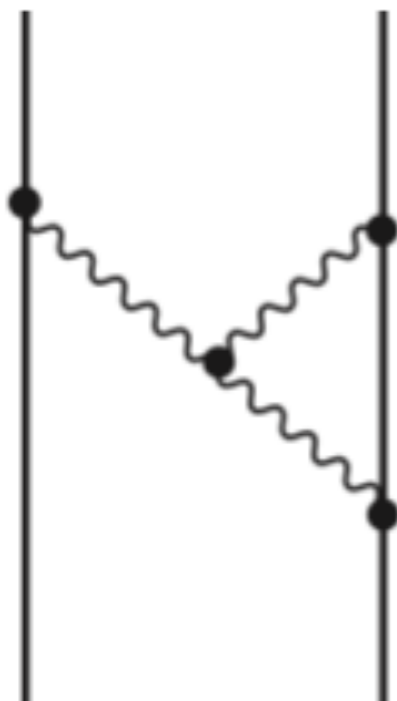
From spacetime spin four-vectors s_1, s_2 to spatial spin three-vectors S_1, S_2

Bini-Damour '17,'18 (see also Vines'18, Vines-Steinhoff-Buonanno'18, Guevara-Ochirov-Vines'18)

spin-scattering holonomy

$$R_1 = B_\eta(\Lambda_1 u_1^- \rightarrow U^{\text{as}}) \Lambda_1 [B_\eta(u_1^- \rightarrow U^{\text{as}})]^{-1}.$$

Using the results of Bel-Damour-Deruelle-Ibanez-Martin'81 for the 2PM metric, one gets the 2PM-accurate value of the integrated spin rotation (spin holonomy)



$$\begin{aligned} \theta_1 = & -\frac{2}{hj\sqrt{\gamma^2-1}} [\gamma X_2 + (2\gamma^2-1)(X_1-h)] \\ & + \frac{\pi}{4h^2j^2} [-3(5\gamma^2-1)(X_1-h) - 6\gamma X_2 \\ & + \gamma(5\gamma^2-3)X_1X_2]. \end{aligned} \quad (E)$$

EOB transcription of the 2PM-accurate spin-rotation

energy spin-gauge
instead of DJS gauge

$$g_S = g_S^{1\text{PM}}(H_{\text{eff}}) + g_S^{2\text{PM}}(H_{\text{eff}})u + O(u^2),$$

$$g_{S^*} = g_{S^*}^{1\text{PM}}(H_{\text{eff}}) + g_{S^*}^{2\text{PM}}(H_{\text{eff}})u + O(u^2),$$

$$\theta_1^{\text{EOB}} = G \int \frac{\mathbf{L}}{R^3} \left(g_S + \frac{m_2}{m_1} g_{S^*} \right) \frac{B}{A} E_{\text{eff}} \frac{dR}{P_R},$$

$$g_S^{1\text{PM}}(\gamma, \nu) = \frac{(2\gamma + 1)(2\gamma + h) - 1}{h(h + 1)\gamma(\gamma + 1)}$$

$$g_{S^*}^{1\text{PM}}(\gamma, \nu) = \frac{1}{h(h + 1)} \left[4 + \frac{h - 1}{\gamma + 1} + \frac{h - 1}{\gamma} \right]$$

$$g_{S^*}^{1\text{PM}}(\gamma, \nu) = \frac{2\gamma + 1}{h\gamma(\gamma + 1)}$$

$$g_{S^*}^{1\text{PM}}(\gamma, \nu) = \frac{1}{h} \left[\frac{1}{\gamma + 1} + \frac{1}{\gamma} \right].$$

$$\gamma = \hat{H}_{\text{eff}} \quad h = \sqrt{1 + 2\nu(\gamma - 1)}$$

$$g_S^{2\text{PM}}(\gamma, \nu) = -\frac{\nu}{\gamma(\gamma + 1)^2 h^2 (h + 1)^2} [2(2\gamma + 1)(5\gamma^2 - 3)h + (\gamma + 1)(35\gamma^3 - 15\gamma^2 - 15\gamma + 3)]$$

$$= \frac{\nu}{h^2 (h + 1)^2} \left[-5(7\gamma + 4h - 10) + \frac{8(3h - 4)}{\gamma + 1} - \frac{4h}{(\gamma + 1)^2} + \frac{3(2h - 1)}{\gamma} \right]$$

$$g_{S^*}^{2\text{PM}}(\gamma, \nu) = -\frac{1}{2\gamma(\gamma + 1)^2 h^2 (h + 1)} [(5\gamma^2 + 6\gamma + 3)(h + 1) + 4\nu(1 + 2\gamma)(5\gamma^2 - 3)]$$

$$= \frac{1}{h^2 (h + 1)} \left[-20\nu + \frac{24\nu - h - 1}{\gamma + 1} + \frac{h + 1 - 4\nu}{(\gamma + 1)^2} - \frac{3h + 1 - 4\nu}{2\gamma} \right]$$

$$= \frac{1}{h^2 (h + 1)} \left[-\frac{20\gamma\nu}{\gamma + 1} + (h + 1 - 4\nu) \left(\frac{1}{(\gamma + 1)^2} - \frac{1}{\gamma + 1} - \frac{3}{2\gamma} \right) \right].$$

Quantum Scattering Amplitudes and 2-body Dynamics

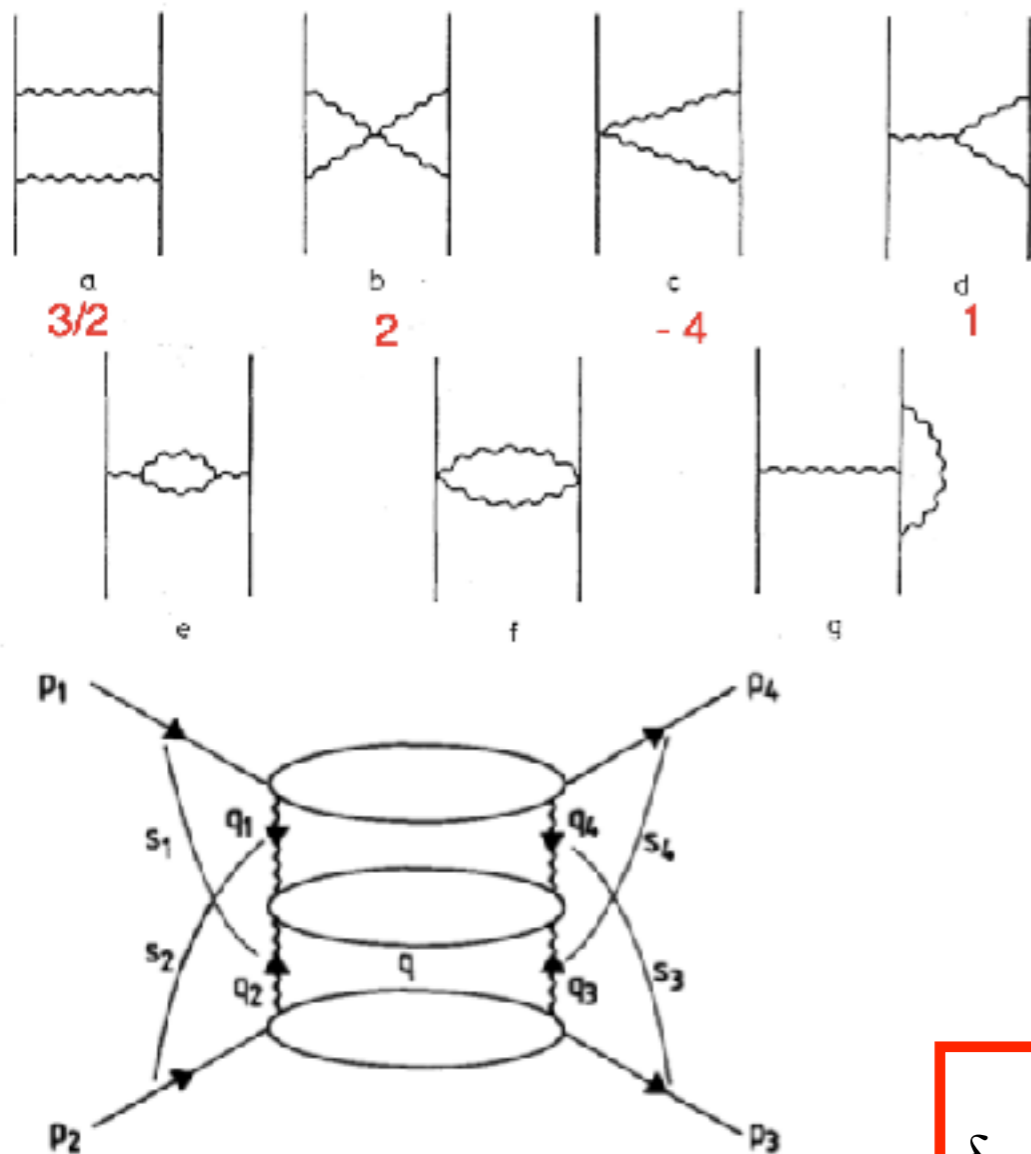


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

- Quantum Scattering Amplitudes → Potential one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [First post-Newtonian approx.],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

Amati-Ciafaloni-Veneziano 1987-2008

Ultra-High-Energy ($s \gg M_{\text{Planck}}^2$)

Four-graviton Scattering at 2 loops

Eikonal phase δ in $D=4$

with one- and two-loop corrections

using the Regge-Gribov approach

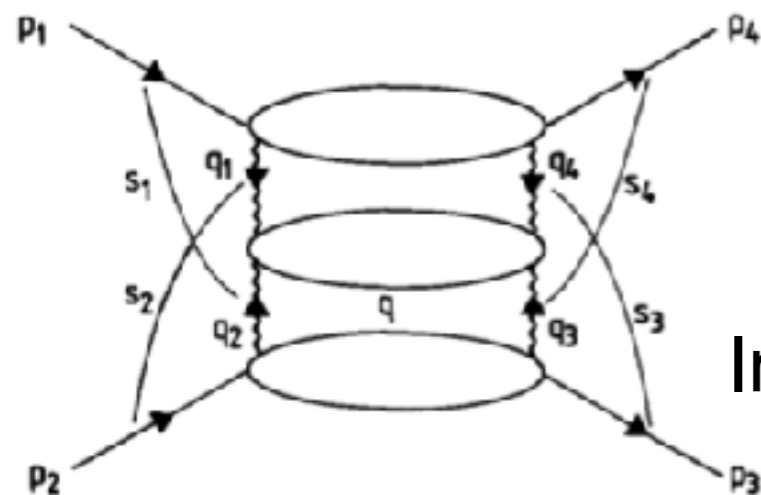
$$\delta = \frac{Gs}{\hbar} \left(\log \left(\frac{L_{IR}}{b} \right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2} \left(1 + \frac{2i}{\pi} \log(\dots) \right) \right)$$

New technique: use EOB as a scattering → Hamiltonian translation device

Progress in gravity amplitudes (Bern, Carrasco et al., Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,...) can be used (Damour '17 Cheung Rothstein Solon '18) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

High-energy limit of 2-body scattering and 2-body dynamics

Using the (eikonal) ultra-high-energy results of Amati-Ciafaloni-Veneziano:
get HE information up to G^4



$$\sin \frac{1}{2} \chi^{\text{ACV HE}} \equiv 2\alpha + (2\alpha)^3 + O(\alpha^5),$$

with $\alpha_{\text{HE}} = \frac{GE_{\text{real}}}{b}$

In HE limit the EOB energy map is such that

$$\alpha = \frac{GME_{\text{eff}}}{J} = \frac{G}{2} \frac{(E_{\text{real}})^2 - m_1^2 - m_2^2}{J} \approx_{\text{HE}} \frac{GE_{\text{real}}}{b}$$

The masses disappear and the HE scattering is equivalent to a null geodesic in the « effective HE metric »

$$ds^2 = -A_{\text{HE}}(u)dT^2 + \frac{dR^2}{1-2u} + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$A_{\text{HE}}(u) = (1 - 2u) \left(1 + \frac{15}{2}u^2 - 3u^3 + \frac{1749}{16}u^4 + O(u^5) \right)$$

2PM 3PM 4PM

Translating quantum scattering amplitudes into classical dynamical information

How to translate a scattering amplitude into a classical Hamiltonian ?

$$\mathcal{M}(s, t) = \mathcal{M}^{(\frac{G}{\hbar})}(s, t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s, t) + \dots$$

$$\mathcal{M}^{(\frac{G}{\hbar})}(s, t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t}$$

NB: The domain of validity of the Born expansion is $GE_1 E_2 / (\hbar v) \ll 1$, while the domain of validity of the classical scattering is $GE_1 E_2 / (\hbar v) \gg 1$!

Proposed quantum/classical transcriptions:

Damour'17: EOB potential $Q(R, E)$

Cheung-Rothstein-Solon'18: different potential $V(R, P^2)$

$$Q^E(u, \mathcal{E}_{\text{eff}}) = u^2 q_2(\mathcal{E}_{\text{eff}}) + u^3 q_3(\mathcal{E}_{\text{eff}}) + u^4 q_4^E(\mathcal{E}_{\text{eff}}) + O(G^5)$$

$$H(\mathbf{P}, \mathbf{X}) = \sqrt{m_1^2 + \mathbf{P}^2} + \sqrt{m_2^2 + \mathbf{P}^2} + V(R, \mathbf{P}^2)$$

$$V(R, \mathbf{P}^2) = G \frac{c_1(\mathbf{P}^2)}{R} + G^2 \frac{c_2(\mathbf{P}^2)}{R^2} + G^3 \frac{c_3(\mathbf{P}^2)}{R^3} + \dots$$

EOB map: **quantize the classical EOB Hamiltonian** dynamics.

2PM, $O(G^2)$ EOB classical mass-shell condition

$$\mathbf{p}^2 = p_\infty^2 + w(r, p_\infty)$$

$$w(r, p_\infty) = \frac{w_1(\gamma)}{r} + \frac{w_2(\gamma)}{r^2} + \frac{w_3(\gamma)}{r^3} + \frac{w_4(\gamma)}{r^4} + \dots$$

$$p_\infty^2 = \hat{\mathcal{E}}_{\text{eff}}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\text{eff}}^2 - 1),$$

$$w_2 = \frac{35\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{2h(\hat{\mathcal{E}}_{\text{eff}})}.$$

Quantized version:

$$-\hat{h}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{r} + \frac{w_2}{r^2} + \frac{w_3}{r^3} + O\left(\frac{1}{r^4}\right) \right] \psi(\mathbf{x})$$

Scattering amplitude for this **potential scattering** computable by higher-order Born approximation

$$\psi_a^+ \underset{r \rightarrow \infty}{\approx} e^{i\mathbf{k}_a \cdot \mathbf{x}} + f_{\mathbf{k}_a}^+(\Omega) \frac{e^{ikr}}{r}$$

$$f_{\mathbf{k}_a}(\hat{\mathbf{k}}_b) = + \frac{1}{4\pi\hat{h}^2} \langle \varphi_b | w | \psi_a^+ \rangle$$

Classical/quantum dictionary: **one-loop**

from EOB $q_2(E) \leftrightarrow w_2(E)$ (classical)

$$f_{\mathbf{k}_a}^{+\text{B1}}(\mathbf{k}_b) = \frac{1}{\hat{\hbar}^2} \left[e^{\delta_c} \frac{w_1}{q^2} + \frac{\pi w_2}{2q} \right],$$

$\mathcal{M}^{G^2} / \mathcal{M}^{G^1}$

$$\frac{f_{(1/q)}^+}{f_{(1/q^2)}^+} = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{2\hat{\mathcal{E}}_{\text{eff}}^2 - 1} \frac{G(m_1 + m_2)\sqrt{-t}}{\hbar} + \mathcal{O}(G^2).$$

OK with one-loop result of Guevara 1706.02314;
and Bjerrum-Bohr et al. 2018

**Breakthrough: Bern et al. 1901.04424: 2-loop amplitude
? gives 3PM $\mathcal{O}(G^3)$ EOB Hamiltonian: $q_3 u^3$**

**higher-
order
Born**

$$f_{k_a}^+(k_b) = \frac{1}{\hat{\hbar}^2} \left[e^{\delta_c} \left(\frac{w_1}{q^2} + \frac{\pi w_2}{2q} \right) + (w_3 \& w_1 w_2) \log q \right],$$

What is known at 3PM ($G^3=2$ -loops) ?

$$q_3(\gamma, \nu) = A(\gamma) + \frac{B(\gamma)}{h(\gamma, \nu)} + \frac{C(\gamma)}{h^2(\gamma, \nu)}$$

$$= A(\gamma) + \frac{B(\gamma)}{h(\gamma, \nu)} - \frac{A(\gamma) + B(\gamma)}{h^2(\gamma, \nu)}$$

$$B(\gamma) = -\hat{q}_3^{(1)}(\gamma) = +\frac{3(2\gamma^2 - 1)(5\gamma^2 - 1)}{2(\gamma^2 - 1)}$$

$$= +\frac{3(1 + 2p_{\text{eob}}^2)(4 + 5p_{\text{eob}}^2)}{2p_{\text{eob}}^2}$$

$$= \frac{6 + \frac{39}{2}p_{\text{eob}}^2 + 15p_{\text{eob}}^4}{p_{\text{eob}}^2}$$

$$A(\gamma) = A^{4\text{PN}}(\gamma) + O(p_{\text{eob}}^4)$$

$$A^{4\text{PN}}(\gamma) = \frac{2 + \frac{37}{2}p_{\text{eob}}^2 + \frac{117}{10}p_{\text{eob}}^4}{p_{\text{eob}}^2}$$

$$p_{\text{eob}}^2 \equiv \gamma^2 - 1$$

$$A(\gamma) \stackrel{\text{HE}}{=} a_3 p_{\text{eob}}^2 \quad a_{-3}^{\text{ACV}} = -3$$

Bern
et al 19.

$$A^{B7\text{PN}}(\gamma) = \frac{1}{p_{\text{eob}}^2} \left(2 + \frac{37}{2}p_{\text{eob}}^2 + \frac{117}{10}p_{\text{eob}}^4 + \frac{219}{140}p_{\text{eob}}^6 \right. \\ \left. - \frac{7079}{10080}p_{\text{eob}}^8 + \frac{989}{2240}p_{\text{eob}}^{10} + O(p_{\text{eob}}^{12}) \right).$$

Bini-
Damour-
Geralico'19

$$A^{5\text{PN}}(\gamma) = \frac{1}{p_{\text{eob}}^2} \left(2 + \frac{37}{2}p_{\text{eob}}^2 + \frac{117}{10}p_{\text{eob}}^4 + \frac{219}{140}p_{\text{eob}}^6 \right)$$

Bern
et al 19.+
Antonelli
et al.19

$$q_{3\text{PM}} = -\frac{2\hat{H}_S^2 - 1}{\hat{H}_S^2 - 1} q_{2\text{PM}} + \frac{4}{3}\nu\hat{H}_S \frac{14\hat{H}_S^2 + 25}{1 + 2\nu(\hat{H}_S - 1)}$$

$$+ \frac{8\nu}{\sqrt{\hat{H}_S^2 - 1}} \frac{4\hat{H}_S^4 - 12\hat{H}_S^2 - 3}{1 + 2\nu(\hat{H}_S - 1)} \sinh^{-1} \sqrt{\frac{\hat{H}_S - 1}{2}}.$$

? incompatible
with HE behavior ₃₈

From 2-loop amplitude to 3PM $O(G^3)$ EOB Hamiltonian i.e. $q_3 u^3$

$$\hat{Q} = u^2 q_2(\hat{H}_{\text{eff}}^{\text{Schw}}) + u^3 q_3(\hat{H}_{\text{eff}}^{\text{Schw}}) + O(u^4),$$

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \dots$$

Potential scattering

$$-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\bar{x}) = W\left(\frac{1}{\bar{r}}\right) \psi(\bar{x})$$

$$\bar{W}(\bar{u}) = w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \dots$$

$$p_\infty^2 = \hat{\mathcal{E}}_{\text{eff}}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\text{eff}}^2 - 1),$$

$$w_2 = \frac{3}{2} \frac{5\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{h(\hat{\mathcal{E}}_{\text{eff}})}.$$

$$w_3(\hat{\mathcal{E}}_{\text{eff}}) = -q_3(\hat{\mathcal{E}}_{\text{eff}}) \ \& \ q_2(\hat{\mathcal{E}}_{\text{eff}}) \ \& \ 1$$

3PM

Born expansion of the scattering amplitude

(Coulomb + second-order in w_1 w_2)

$$f_{k_a}^+(k_b) = \frac{1}{\hat{\hbar}^2} \left[e^{\delta_c} \frac{w_1}{q^2} + \frac{\pi}{2} \frac{w_2}{q} + (w_3 \ \& \ w_1 w_2) \log q \right],$$

Checks:

4PN

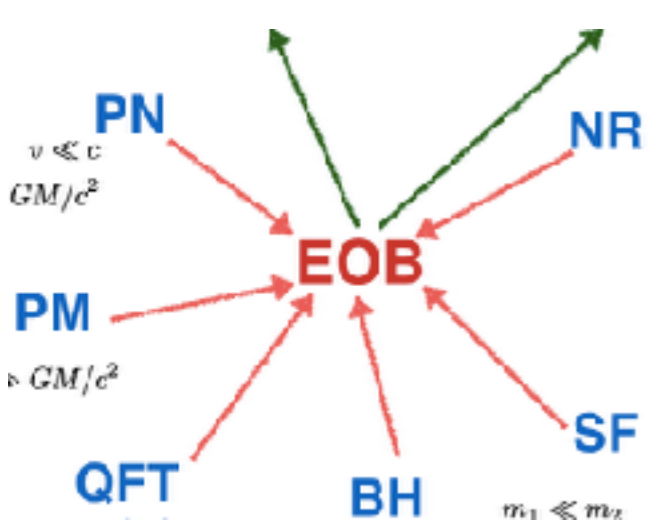
directly determines $q_3(E_{\text{eff}})$

$$q_3^{\text{PN}}(\hat{H}_{\text{Schw}}; \nu) = 5\nu + \frac{1}{4}(108\nu - 23\nu^2)(\hat{H}_{\text{Schw}}^2 - 1) + 4\text{PN}$$

and EOB[ACV]

$$q_3(\hat{\mathcal{E}}_{\text{eff}}; \nu) \stackrel{\text{HE}}{=} -3 \hat{\mathcal{E}}_{\text{eff}}^2$$

Summary



EOB theory (Hamiltonian + Waveform) is directly used by LIGO-Virgo to define the many needed accurate templates

The EOB formulation of 2-body dynamics is a useful tool for transcribing classical and quantum scattering information into bound-state information.

The classical one-loop (G^2) scattering has been transcribed in EOB theory thereby giving new vistas on high-energy gravitational interactions.

The Amati-Ciafaloni-Veneziano 2-loop HE result has been transcribed in EOB theory.

A quantum/classical dictionary has been established.

The EOB-based dictionary allows for an easy and simple translation of two-loop quantum scattering amplitude of gravitationally interacting into a 3PM EOB Hamiltonian.

Serious doubts about the correctness of the conjectured resummed 3PM (2-loop) result of Bern-Cheung-Roiban-Shen-Solon-Zeng'19

EOB offers also a useful framework to transcribe spin effects from classical to quantum

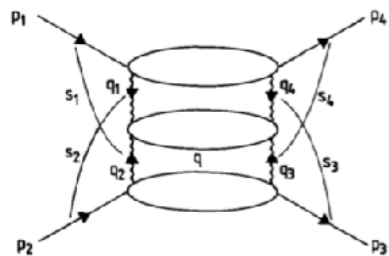
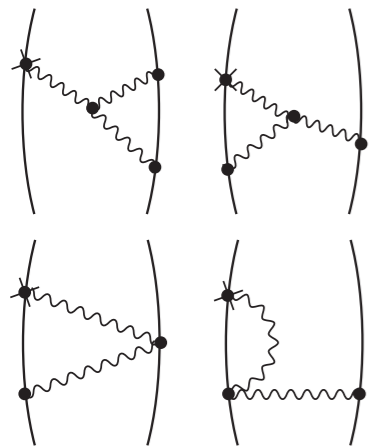


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

$$-\hbar^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_{\infty}^2 + \frac{w_1}{r} + \frac{w_2}{r^2} + O\left(\frac{1}{r^3}\right) \right] \psi(\mathbf{x}).$$