Modular properties of Closed Superstring Theory Scattering Amplitudes

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- **Overview of higher-derivative interactions in closed superstring theory**

  NON-PERTURBATIVE FEATURES – S-DUALITY IN SUPERSTRING THEORY:

  Connects perturbative with non-perturbative effects.

  Constraints imposed by SUSY, Duality

  Older work with Pierre Vanhove, Sav Sethi, Michael Gutperle, Anirban Basu, ….

- **SL(2,Z) modular forms and U(1)-violation in IIB superstring**

  with CONGKAO WEN Arxiv: 1904.13394

  First-order differential relations between coefficients in low energy expansion, which imply Laplace eigenvalue equations for low order terms.

  Modular forms for coefficients of n-point maximal U(1)-violating interactions

- **Elegant connections with graviton scattering amplitudes**

MOTIVATION: HOLOGRAPHIC CONNECTION OF TYPE IIB SUPERSTRING AMPLITUDES WITH GAUGE-INVARIANT CORRELATION FUNCTIONS OF N=4 SUSY YANG-MILLS

Montonen-Olive SL(2,Z) duality of N=4 SUSY Yang-Mills $\Longleftrightarrow$ **SL(2,Z) S-duality of type IIB superstring**

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical supergravity

EINSTEIN-HILBERT

$\alpha' = \ell_s^2$

ℓ_s - STRING LENGTH SCALE

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$ Interactions of other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$e^{-\phi} = \dfrac{1}{g_s}$ STRING COUPLING CONSTANT

compactify space-time to dimensions D < 10

- HIGHER ORDER TERMS: (maximal supersymmetry)

$$\frac{1}{\alpha'}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\ldots)\,R^4 + \ldots$$

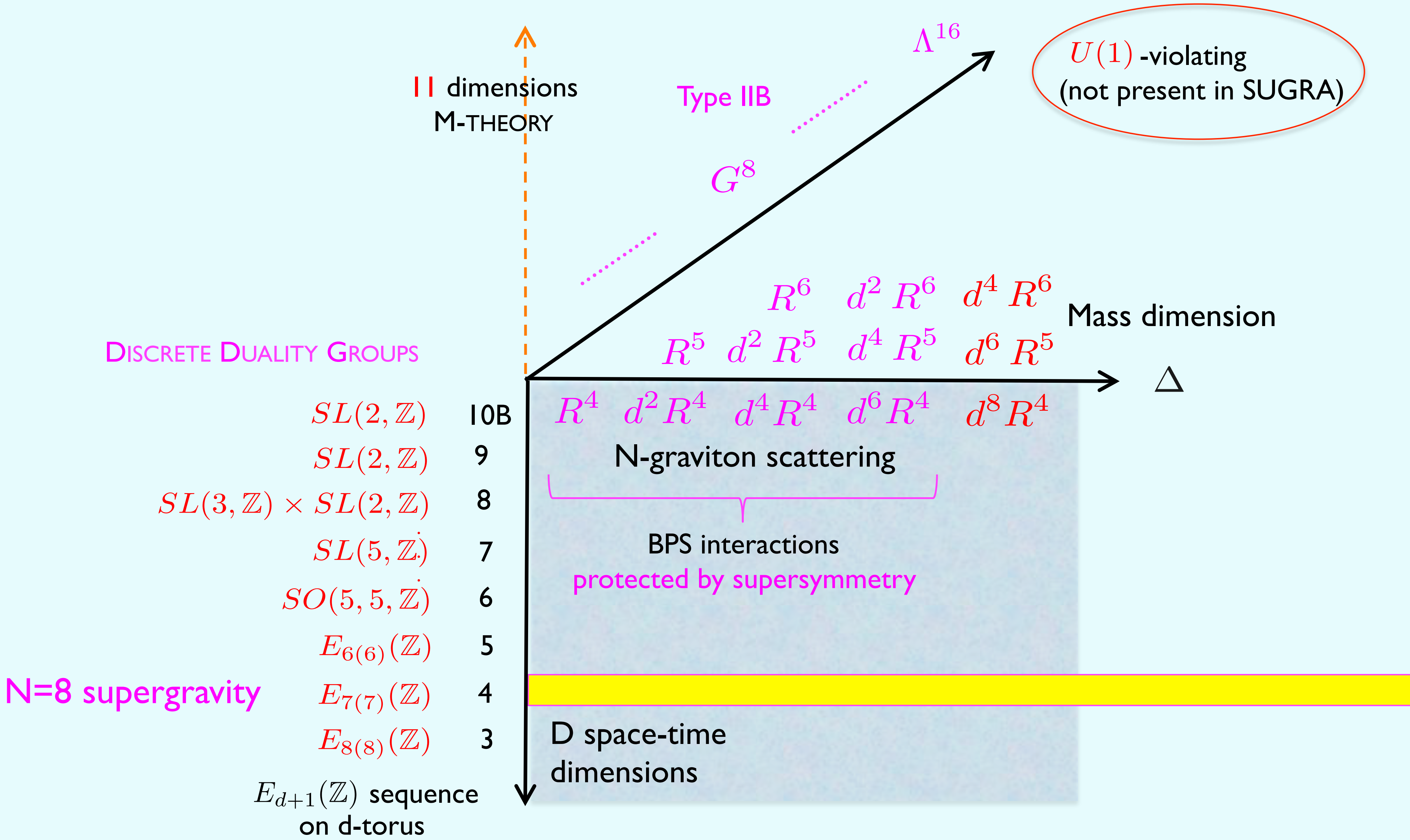
Coefficient depends on moduli (scalar fields).
Constrained by **S-DUALITY**

- Expansion in powers of $\alpha' R\,,\quad \alpha' D^2\,,\quad \ldots$

- Double expansion – perturbative expansion in powers of $g = e^{-\phi}$

The Low Energy Expansion of (Type IIB) String Theory

Higher derivative corrections to Einstein theory



Scalar Fields (Moduli) and S-Duality

Supergravity (low energy limit of string theory):

Scalar fields parameterize a symmetric space $G(\mathbb{R})/K(\mathbb{R})$ (Cremmer, Julia)

Maximal compact subgroup

String theory:

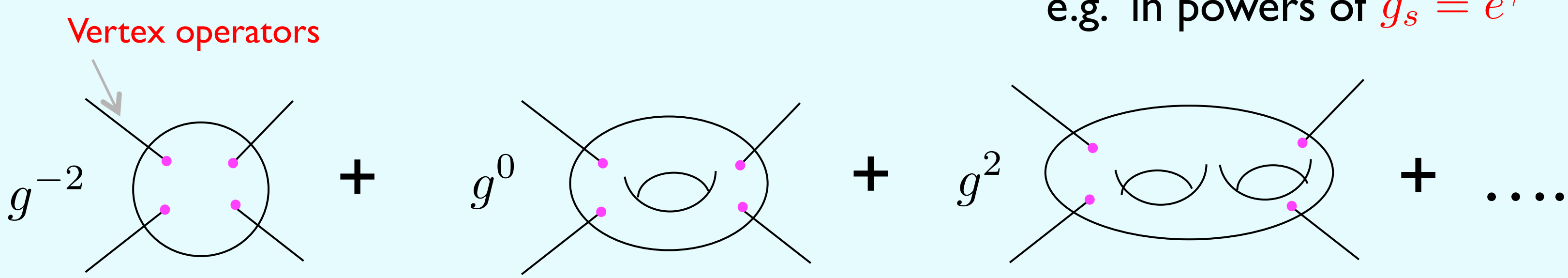
Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

Duality group $G(\mathbb{Z})$

Only a discrete arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory

String perturbation theory: Expansion around boundary of moduli space

e.g. in powers of $g_s = e^{\phi}$



$SL(2,\mathbb{Z})$ $Sp(4,\mathbb{Z})$ "$Sp(2h,\mathbb{Z})$"

World-sheet torus

How powerful are the constraints imposed by (maximal) SUSY and Duality ??

Investigate the exact moduli dependence of low lying terms in the low energy expansion.

Duality relates different regions of moduli space –
Connects perturbative and non-perturbative features in a highly nontrivial manner.

Consider simplest example:

10-dimensional Type IIB - maximal supersymmetry

One complex modulus $\tau = \tau_1 + i\tau_2$

inverse string coupling constant $\longrightarrow$ $\tau_2 = \frac{1}{g} = e^{-\phi}$

Duality group $SL(2,\mathbb{Z})$ $\tau \to \frac{a\tau + b}{c\tau + d}$ $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$

Relates strong and weak coupling.

$SL(2,\mathbb{R})$ Transformations of massless Type IIB fields

- Fermions transform under the local $U(1)$ but not the global $SL(2,\mathbb{R})$.
  Other fields transform under $SL(2,\mathbb{R})$.

$SL(2,\mathbb{R})$ matrix $V^{\alpha}_{\ i}$ transforms by $V^{\alpha}_{\ i} \to U^{\alpha}_{\ \beta}\, V^{\beta}_{\ i}\, R^{i}_{\ j}$

$SL(2,\mathbb{R})$ $O(2) \sim U(1)$

- Fix $U(1)$ gauge $\phi = 0$ - embed the $U(1)$ in $SL(2,\mathbb{R})$

$e^{2i\phi} = \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)$

- A $SL(2,\mathbb{R})$ transformation induces a compensating $U(1)$ transformation to preserve gauge condition.

$U(1)$ charge of any field $\Phi = q_\Phi$ (c.f. Schwarz)

Scalar bosons $\quad P_\mu = i\frac{\partial_\mu \tau}{2\tau_2} \quad q_P = -2 \qquad \bar{P}_\mu = -i\frac{\partial_\mu \bar{\tau}}{2\tau_2} \quad q_{\bar{P}} = 2$

Antisymmetric tensors $\quad G = V^{\alpha}_{+}\,\mathcal{F}_\alpha \quad q_G = -1 \qquad \bar{G} = V^{\alpha}_{-}\,\mathcal{F}_\alpha \quad q_{\bar{G}} = 1$

SL(2) doublet $\mathcal{F}^\alpha = \begin{pmatrix} dB_2 \\ dC^{(2)} \end{pmatrix}$ $\qquad$ SL(2) singlet $G,\ \bar{G}$

Neutral bosons $\quad dC^{(4)},\ R \quad q_{\mathcal{F}_5} = q_R = 0$

Fermions $\quad \Lambda \quad q_\Lambda = -\frac{3}{2} \qquad \bar{\Lambda} \quad q_{\bar{\Lambda}} = \frac{3}{2} \qquad \psi_\mu \quad q_\psi = -\frac{1}{2} \qquad \bar{\psi}_\mu \quad q_{\bar{\psi}} = \frac{1}{2}$

Note: Chiral U(1) Anomaly in Type IIB Supergravity in D=10 Dimensions

The anomaly cancelling term breaks $SL(2,\mathbb{R})$ to $SL(2,\mathbb{Z})$

Systematics of U(1) violation

Consider a *linearised* constrained on-shell SCALAR CHIRAL ON-SHELL SUPERFIELD describing fluctuations around $\tau = \tau_0$. Function of a single 16-component Grassman spinor, θ.

(Howe, West)

$$\Phi(x,\theta) = \delta\tau + \theta\,\Lambda + \theta^2\,G + \theta^3\,d\psi \;+ \theta^4\,(R^4 + dF_5) + \theta^5\,d^2\psi^* + \theta^6\,d^2\bar{G} + \theta^7\,d^3\Lambda^* + \theta^8\,d^4\bar{\tau}$$

U(1)-charge of superfield $= -2$ U(1)-charge of $\theta = -\frac{1}{2}$

Linearised action $$\int d^{16}\theta\, F[\tau_0 + \Phi(x,\theta)] \;= \int d^{16}\theta \sum_n \frac{\partial^n F(\tau_0)}{\partial\tau_0^n}\,[\Phi(x,\theta)]^n$$

$$= F_{(4)}(\tau_0)\,R^4 + F_{(5)}(\tau_0)\,G^2\,R^3 + \cdots + F_{(16)}(\tau_0)\,\Lambda^{16} \qquad F_{(n)}(\tau_0) = \frac{\partial^n F(\tau_0)}{\partial\tau_0^n}$$

- U(1) VIOLATION FOR N-POINT FUNCTIONS: $q = -2(n-4)$ All four-point functions conserve U(1)

  Maximal U(1) violation in

- These 8-derivative interactions are $\frac{1}{2} - BPS$

  $\frac{1}{4} - BPS$ $\frac{1}{8} - BPS$

- More generally consider derivatives on these interactions - e.g. $d^4 R^4$, $d^6 R^4$

- Note for future reference that $F_{\Lambda^{16}} = \dfrac{\partial^{12} F_{R^4}}{\partial\tau_0^{12}}$

Higher Derivative SL(2,Z)-Covariant Action

The linearised interactions fit into a $SL(2,\mathbb{Z})$ - invariant action of the form

$$\kappa = (\alpha')^2 g \qquad S_n^p = (\kappa)^{\frac{p-1}{2}} \int d^{10}x\, e\, F_{w\,i}^{(p)}(\tau)\, d_{(i)}^{2p}\, \mathcal{P}_n(\{\Phi\})$$

Monomial in n fields

$R^4 \quad p = 0$
$d^4 R^4 \quad p = 2$
$d^6 R^4 \quad p = 3$

Degeneracy in kinematic factors

$q = -2(n-4)$
$= -2w$

- Derivatives $d_{(i)}^{2p}$ (contractions suppressed) explicit in amplitude calculations

  e.g. for $n = 4,\ p = 3 \quad d^6 R^4 \sim (s^3 + t^3 + u^3)\, R^4$

- **Degeneracy** first arises for n=4, p=6; n=5, p=4; **n=6, p=3** e.g. $d^6 G^4 R^2$

- The quantity $\mathcal{P}_n(\{\Phi\})$ is the product of n fields in linearised approximation with $q = -2(n-4)$

- Since $\mathcal{P}_n(\{\Phi\})$ carries a non-zero U(1) charge, the moduli-dependent coefficient $F_{w\,i}^{(p)}(\tau)$ must transform with a compensating charge.

Non-holomorphic modular form

modular weight w

- The complete nonlinear action is not known - even in the p=0 case (1/2-BPS).

  although it is known for backgrounds in which certain bosonic fields vanish

Non-Holomorphic Modular Forms

Consider a $SL(2,\mathbb{Z})$ transformation $\tau \to \dfrac{a\tau+b}{c\tau+d}$ $\quad a,b,c,d \in \mathbb{Z} \quad ad-bc=1$

holomorphic anti-holomorphic

A NON-HOLOMORPHIC MODULAR FORM with weight (w, w') transforms as

$$f^{(w,w')}(\tau) \to (c\tau+d)^w\,(c\bar\tau+d)^{w'}\,f^{(w,w')}(\tau)$$

So if $w' = -w$

$$f^{(w,-w)}(\tau) \to \left(\frac{c\tau+d}{c\bar\tau+d}\right)^w f^{(w,-w)}(\tau)$$

$e^{2iw\phi}$ $\quad \phi = \dfrac{i}{2}\log\left(\dfrac{c\bar\tau+d)}{c\tau+d}\right)$

Transforms with phase – U(1) charge $q = 2w$

COVARIANT DERIVATIVES: $\quad \mathcal{D}_w = i\tau_2\dfrac{\partial}{\partial\tau} + \dfrac{w}{2} \qquad \bar{D}_{w'} = -i\tau_2\dfrac{\partial}{\partial\bar\tau} + \dfrac{w'}{2}$

$$\mathcal{D}_w\, f^{(w,-w)} = f^{(w+1,-w-1)}$$

Increases the U(1) charge

$$\bar{\mathcal{D}}_w\, f^{(w,-w)} = f^{(w-1,-w+1)}$$

Decreases the U(1) charge

Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

- $SL(2,\mathbb{Z})$ invariant (generalises to higher rank duality groups) - weight $(0,0)$ form

- Solution of Laplace eigenvalue eqn.

$$\Delta_\tau\, E(s,\tau) = s(s-1)\, E(s,\tau)$$

- Fourier series $\quad E(s,\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\tau_2)\, \cos(2\pi i k \tau_1)$

- Zero mode $\quad k=0$ - TWO POWER-BEHAVED TERMS (perturbative) :

$$\mathcal{F}_0 = 2\zeta(2s)\,\tau_2^s + \frac{2\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\,\zeta(2s-1)}{\Gamma(s)}\,\tau_2^{1-s}$$

- Non-zero modes $\quad k>0$ - D-INSTANTON SUM

$$\mathcal{F}_k = \frac{4\pi^s}{\Gamma(s)}\,|k|^{s-\frac{1}{2}}\sigma_{2s-1}\,\tau_2^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi k\tau_2)$$

$$\sim \frac{2\pi^{s-\frac{1}{2}}}{\Gamma(s)}\,|k|^{s-1}\sigma_{2s-1}(k)\,e^{-2\pi k\tau_2}\,\left(1+O(\tau_2^{-1})\right)$$

divisor sum

$$\sigma_n(k) = \sum_{p|k} p^n$$

LOW ORDER INTERACTION COEFFICIENTS

for U(1)-conserving four-point amplitudes

Laplace equations motivated by supersymmetry and various dualities

$\Delta = \tau_2^2\,(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) = 4\partial_\tau\,\partial_{\bar\tau}$

$(\alpha')^{-1}\,R^4$ $\quad \left(\Delta - \frac{3}{4}\right) F_0^{(0)}(\tau) = 0 \qquad F_0^{(0)}(\tau) = E(\tfrac{3}{2},\tau)$

$$g^{-\frac{1}{2}}\,E(\tfrac{3}{2},\tau) = 2\zeta(3)\,g^{-2} + 4\zeta(2)\,g^0 + \sum_{k=0}(\dots)\,\sigma_2(k)\,e^{-2\pi|k|\tau_2}e^{2\pi ik\tau_1}$$

Perturbative terms: tree-level genus-one D-instantons

NON-RENORMALISATION BEYOND 1 LOOP FOR R^4 $\quad \frac{1}{2} - BPS$

$\alpha'\,d^4\,R^4$ $\quad \left(\Delta - \frac{15}{4}\right) F_0^{(2)}(\tau) = 0 \qquad F_0^{(0)}(\tau) = E(\tfrac{5}{2},\tau)$

$$g^{\frac{1}{2}}\,E(\tfrac{5}{2},\tau) = 2\zeta(5)\,g^{-2} + \frac{4}{3}\zeta(4)\,g^2 + \sum_{k=0}(\dots)\,\sigma_4(k)\,e^{-2\pi|k|\tau_2}e^{2\pi ik\tau_1}$$

Perturbative terms: tree-level genus-two D-instantons

NON-RENORMALISATION BEYOND 2 LOOPS FOR $d^4\,R^4$ $\quad \frac{1}{4} - BPS$

$(\alpha')^2 \, d^6 \, R^4$ $$F_0^{(3)}(\tau) = \mathcal{E}_0^{(3)}(\tau)$$

NOT Eisenstein series but satisfies INHOMOGENEOUS Laplace equation

$$(\Delta - 12)\, \mathcal{E}_0^{(3)}(\tau) = -E(\tfrac{3}{2}, \tau) E(\tfrac{3}{2}, \tau) \longleftarrow$$

The square of the coefficient of R^4

THE SOLUTION OF THIS EQUATION HAS SOME WEIRD AND WONDERFUL FEATURES.

ZERO MODE OF SOLUTION (zero net D-instanton number):

$$q\, \mathcal{E}_0^{(3)}(\tau)\Big|_{\substack{zero\\ modes}} = \frac{2}{3}\zeta(3)^2\, g^{-2} + \frac{4}{3}\zeta(2)\,\zeta(3)\, g^0 + 4\zeta(4)\, g^2 + \frac{4}{27}\zeta(6)\, g^4 + \sum_k c_k e^{-4\pi k/g}$$

GENUS ZERO GENUS ONE GENUS TWO GENUS THREE SUM OF D-INSTANTONS

PRECISE AGREEMENT WITH EXPLICIT PERTURBATIVE STRING THEORY MULTI-LOOP CALCULATIONS

[PARENTHETICAL COMMENT: THE NON-RENORMALISATION STATEMENTS IN MAXIMAL SUPERGRAVITY ARE IN AGREEMENT WITH THESE STRING THEORY RESULTS.]

First-Order Equations for U(1)-Violating Coefficients

The coefficient of a term violating the U(1) charge by $q = -2(n-4) = -2w$ units is given by

$$F^{(0)}_{n-4}(\tau) = c^{(0)}_n \, E_w(\tfrac{3}{2},\tau) \qquad\qquad E_0(s,\tau) \equiv E(s,\tau)$$

Non-holomorphic Eisenstein modular forms

Eisenstein series with holomorphic/anti-holomorphic weights $(w,-w)$ defined by

First-order equations $$E_{w+1}(s,\tau) = \frac{s+w}{2}\,\mathcal{D}_w E_w(\tau)$$ (arbitrary normalisation)

so $$E_w(s,\tau) = \frac{2^w (s-1)!}{s+w-1)!}\,\mathcal{D}_{w-1}\ldots\mathcal{D}_0\,E_0(s,\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{m+n\bar\tau}{m+n\tau}\right)^w \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

Likewise, $$E_{w-1}(s,\tau) = \frac{s-w}{2}\,\bar{\mathcal{D}}_w E_w(\tau)$$

Laplace operators $$\Delta^w_+ = 4\,\bar{\mathcal{D}}_{w+1}\mathcal{D}_w \qquad \Delta^w_- = 4\,\mathcal{D}_{w-1}\bar{\mathcal{D}}_w \qquad \Delta^w_+ - \Delta^w_- = -2w$$

Laplace equations $$\Delta_- E_w = (s(s-1) - w(w-1))\,E_w$$

e,g, $$E_w(\tfrac{3}{2},\tau) = 2\zeta(3)\,\tau_2^{\frac{3}{2}} + \frac{4\zeta(2)}{1-4w^2}\,\tau_2^{-\frac{1}{2}} + \sum_{K=1}^{\infty}\left(C_{K,w}(\tau_2)e^{2\pi i K\tau_1} + C_{K,-w}(\tau_2)e^{-2\pi i K\tau_1}\right)$$

$\tau_2 = 1/g$ tree-level genus-one D-instantons anti-D-instantons

½-BPS and ¼-BPS U(1)-Violating Coefficients

Supersymmetry together with S-duality:

$$
\begin{array}{llllll}
n = 4 - w & 4 & 5 & 6 & 8 & 12 \\
(\alpha')^{-1} : & R^4 & G^2 R^3 & G^4 R^2 \ \ldots & G^8 & \ldots \Lambda^{16}
\end{array}
$$

$$F^{(0)}_{n-4}(\tau) = c^{(0)}_n \, E_w(\tfrac{3}{2}, \tau)$$

$$
\alpha' : \quad d^4 R^4 \quad d^4 G^2 R^3 \quad d^4 G^4 R^2 \ \ldots \ d^4 G^8 \ \ldots d^4 \Lambda^{16}
$$

$$F^{(0)}_{n-4}(\tau) = c^{(2)}_n \, E_w(\tfrac{5}{2}, \tau)$$

- Satisfy sequence of Laplace eigenvalue equations.
- Coefficients determined by amplitude analysis

1/8-BPS U(1)-VIOLATING COEFFICIENTS

$n = 4 + w$ 4   5   6   8   12

$$(\alpha')^2 : \quad d^6_{(i)} R^4 \quad d^6_{(i)} G^2 R^3 \quad \underbrace{d^6_{(i)} G^4 R^2 \ldots d^6_{(i)} G^8 \ldots d^6_{(i)} \Lambda^{16}}_{\text{TWO INDEPENDENT KINEMATIC STRUCTURES}}$$

$$F^{(3)}_{n-4,i}(\tau) = c^{(3)}_{n,i}\, \mathcal{E}^{(3)}_w(\tau)$$

$n = 4 + w$

$n = 6$
$w = 2$

$$d^6_{(1)} \sim \sum_{i<j} s^3_{ij} + \frac{3}{8} \sum_{i<j<k} s^3_{ijk}$$

Tree-level contribution

$$d^6_{(2)} \sim \sum_{i<j} s^3_{ij} - \frac{1}{2} \sum_{i<j<k} s^3_{ijk}$$

Does not contribute at tree-level

RECALL W=0 CASE: $$4\,\bar{\mathcal{D}}\mathcal{D}\,\mathcal{E}^{(3)}_0 = 12\mathcal{E}^{(3)}_0 - (E_0(\tfrac{3}{2}))^2$$

$\Delta = \Delta^0_+ = \Delta^0_-$

CONSIDER W=1 CASE: Define: $\mathcal{E}^{(3)}_1 = 2\mathcal{D}\mathcal{E}^{(3)}_0$ **FIRST-ORDER EQUATION**

Apply $\mathcal{D}$ to $w = 0$ equation, $4\,\mathcal{D}\bar{\mathcal{D}}(\mathcal{D}\,\mathcal{E}^{(3)}_0) = 12(\mathcal{D}\mathcal{E}^{(3)}_0) - \mathcal{D}(E_0(\tfrac{3}{2}))^2$

$$\longrightarrow \quad \Delta_-\mathcal{E}^{(3)}_1 = 12\mathcal{E}^{(3)}_1 - 3\,E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})$$

$w = 1$ **LAPLACE EQUATION**

Applying $\bar{\mathcal{D}}$ and requiring consistency with w=0 Laplace equation leads to

$$\longrightarrow \quad \bar{\mathcal{D}}\mathcal{E}^{(3)}_1 = \mathcal{E}^{(3)}_0 - \frac{1}{12}(E_0(\tfrac{3}{2}))^2$$

INHOMOGENEOUS
FIRST-ORDER EQUATION

1/8-BPS COEFFICIENTS - THE w=2, n=6 CASE

$i=1$

$$\mathcal{E}^{(3)}_{2,i}\, d^6_{(i)}\, (G^4R^2 + \Lambda^8 R^2 + \dots)$$

$i = 1.2$ Labels distinct kinematic structures – motivated by amplitude analysis

The factor $\mathcal{E}^{(3)}_{2,1}(\tau)\, d^6_{(1)}$ contains the tree-level contribution

Define: $\mathcal{E}^{(3)}_{2,1} = 2\mathcal{D}\mathcal{E}^{(3)}_1$ Then consistency with $\mathcal{E}^{(3)}_1$ equation

Leads to **LAPLACE EQUATION** $\Delta_-\mathcal{E}^{(3)}_{2,1} = 10\mathcal{E}^{(3)}_{2,1} - \frac{15}{2}\left(E_0(\tfrac{3}{2})\, E_2(\tfrac{3}{2}) + \frac{3}{5} E_1(\tfrac{3}{2})\, E_1(\tfrac{3}{2})\right)$

and **FIRST-ORDER EQUATION** $\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} = 5\mathcal{D}\mathcal{E}^{(3)}_1 - \frac{3}{2} E_1(\tfrac{3}{2})\, E_0(\tfrac{3}{2})$

$i=2$

The factor $\mathcal{E}^{(3)}_{2,2}(\tau)\, d^6_{(2)}$ does **not** have a tree-level contribution

FIRST-ORDER EQUATION $\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} = a(\mathcal{E}^{(3)}_{2,1} - \frac{1}{2} E_1(\tfrac{3}{2})\, E_1(\tfrac{3}{2}))$

Tree-level term cancels in this combination Leading term from one-loop contribution.

Fix the constant a by one-loop calculation.

$$\mathcal{E}^{(3)}_{2,2} = \frac{a}{5}(\mathcal{E}^{(3)}_{2,1} - 2E_1(\tfrac{3}{2})\, E_1(\tfrac{3}{2}))$$

LAPLACE EQUATION $\Delta_-\mathcal{E}^{(3)}_{2,2} = 10\mathcal{E}^{(3)}_{2,2} - \frac{5a}{12}\left(E_0(\tfrac{3}{2})E_2(\tfrac{3}{2}) - E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})\right).$

1/8-BPS COEFFICIENTS - THE w>2, n>6 CASES

The extension to all terms of the form $\mathcal{E}^{(3)}_{2,i}(\tau)\, d^6_{(i)}\, \mathcal{P}_n(\{\Phi\})$

SUPERSTRING SCATTERING AMPLITUDES (very sketchy)

Recall $$S_n^p = (\kappa)^{\frac{p-1}{2}} \int d^{10}x\, e\, F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\})$$

(i) Amplitudes with external Φ from $\mathcal{P}_n(\Phi)$

(ii) Amplitudes with external fluctuations of τ

Not covariant

does NOT transform covariantly

$$\hat{\tau} = \frac{i}{2}\frac{\tau - \tau^0}{\tau_2^0}$$

$$F_0(\tau) = F(\tau^0) + 2i\tau_2^0\, \partial_{\tau^0} F(\tau^0)\hat{\tau} - 2(\tau_2^0)^2\, \partial^2_{\tau^0} F(\tau^0)\hat{\tau}^2 + \cdots$$

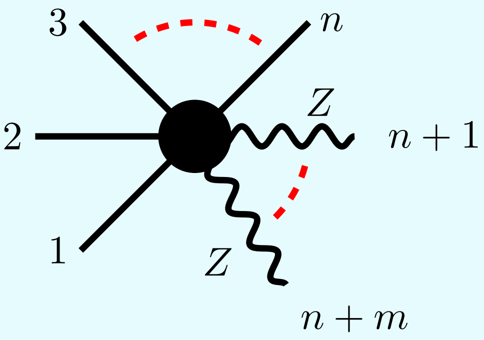
Redefine coordinate $\tau \to Z$ $\qquad Z = \dfrac{\tau - \tau^0}{\tau - \bar{\tau}^0}$

Transforms with U(1) charge = -2 under $SL(2,\mathbb{Z})$

$$F_0(\tau) = \sum_{w=0}^{\infty} 2^w\, \mathcal{D}_{w-1} \ldots \mathcal{D}_0\, F_0(\tau)\Big|_{\tau=\tau^0} Z^w/w! + \cdots$$

Amplitudes with m Zs and n Φ s

$\mathcal{D}^m F^{(p)}_{n-4,i}(\tau^0)$



Super-Amplitudes

Ten-dimensional spinor-helicity formalism encapsulates supersymmetric amplitudes.

$$A_n^{(p)} = \kappa^{\frac{p-1}{2}} F_{n-4}^{(p)}(\tau^0)\, \delta^{16}(Q_n)\, \hat{A}_n^{(p)}(s_{ij}) \qquad \kappa^2 = (\alpha')^4\, \tau_2^{-2}$$

Coupling constant dependence

Dependence on momenta

Enforces supersymmetry – packages the contributions of terms with 16 θ

$\mathcal{P}_n(\Phi)$ and $Z,\ \bar{Z}$ fluctuations

Low-energy expansion of tree-level amplitudes

(thanks to Oliver Schlotterer)

$$\hat{A}_4(s_{ij}) = 2\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\,\zeta(3) + \kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\,\zeta(5)\,\mathcal{O}_4^{(2)} + \frac{2}{3}\,\kappa\,\tau_2^3\,\zeta(3)^2\,\mathcal{O}_4^{(3)} + \cdots$$

$$\hat{A}_5(s_{ij}) = 3\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\zeta(3) + \frac{5}{2}\kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\zeta(5)\,\mathcal{O}_5^{(2)} + 2\,\kappa\,\tau_2^3\zeta(3)^2\,\mathcal{O}_5^{(3)} + \cdots$$

$$\hat{A}_6(s_{ij}) = \frac{15}{2}\,\kappa^{-\frac{1}{2}}\,\tau_2^{\frac{3}{2}}\zeta(3) + \frac{35}{4}\kappa^{\frac{1}{2}}\,\tau_2^{\frac{5}{2}}\zeta(5)\,\mathcal{O}_6^{(2)} + 8\,\kappa\,\tau_2^3\zeta(3)^2\,\mathcal{O}_{6,1}^{(3)} + \cdots\,,$$

where

$$n \le 5 \qquad \mathcal{O}_n^{(2)} = \frac{1}{2}\sum_{1\le<j\le n} s_{ij}^2 \qquad \mathcal{O}_n^{(3)} := \frac{1}{2}\sum_{1\le i<j\le n} s_{ij}^3$$

$$n = 6 \qquad \mathcal{O}_{6,1}^{(3)} = \frac{1}{32}\left(10\sum_{1\le i<j\le 6} s_{ij}^3 + 3\sum_{1\le i<j<k\le 6} s_{ijk}^3\right)$$

Soft Limits

$$A_n(X,\mathbb{Z}_n)\big|_{p_n\to 0} = 2\,\mathcal{D}A_{n-1}(X)\,,$$

More explicitly $$F^{(p)}_{n-4}(\tau^0)\,\mathcal{O}^{(p)}_{n,i}\big|_{p_n\to 0} = 2\,\mathcal{D}F^{(p)}_{n-5}(\tau)\big|_{\tau=\tau^0}\mathcal{O}^{(p)}_{n-1,i}$$

$n\leq 5 \quad p=2,\,3$

$$\mathcal{O}^{(2)}_n = \frac{1}{2}\sum_{1\leq<j\leq n} s^2_{ij} \qquad \mathcal{O}^{(3)}_n = \frac{1}{2}\sum_{1\leq i<j\leq n} s^3_{ij}$$

$$\mathcal{O}^{(2)}_n\big|_{p_n\to 0} = \mathcal{O}^{(2)}_{n-1} \qquad \mathcal{O}^{(3)}_n\big|_{p_n\to 0} = \mathcal{O}^{(3)}_{n-1}$$

$n=6, \quad p=3$

$$\mathcal{O}^{(3)}_{6,1} = \frac{1}{32}\left(10\sum_{1\leq i<j\leq 6} s^3_{ij} + 3\sum_{1\leq i<j<k\leq 6} s^3_{ijk}\right) \qquad \mathcal{O}^{(3)}_{6,2} = \frac{1}{8}\sum_{\text{permutation}} s_{12}s_{34}s_{56}$$

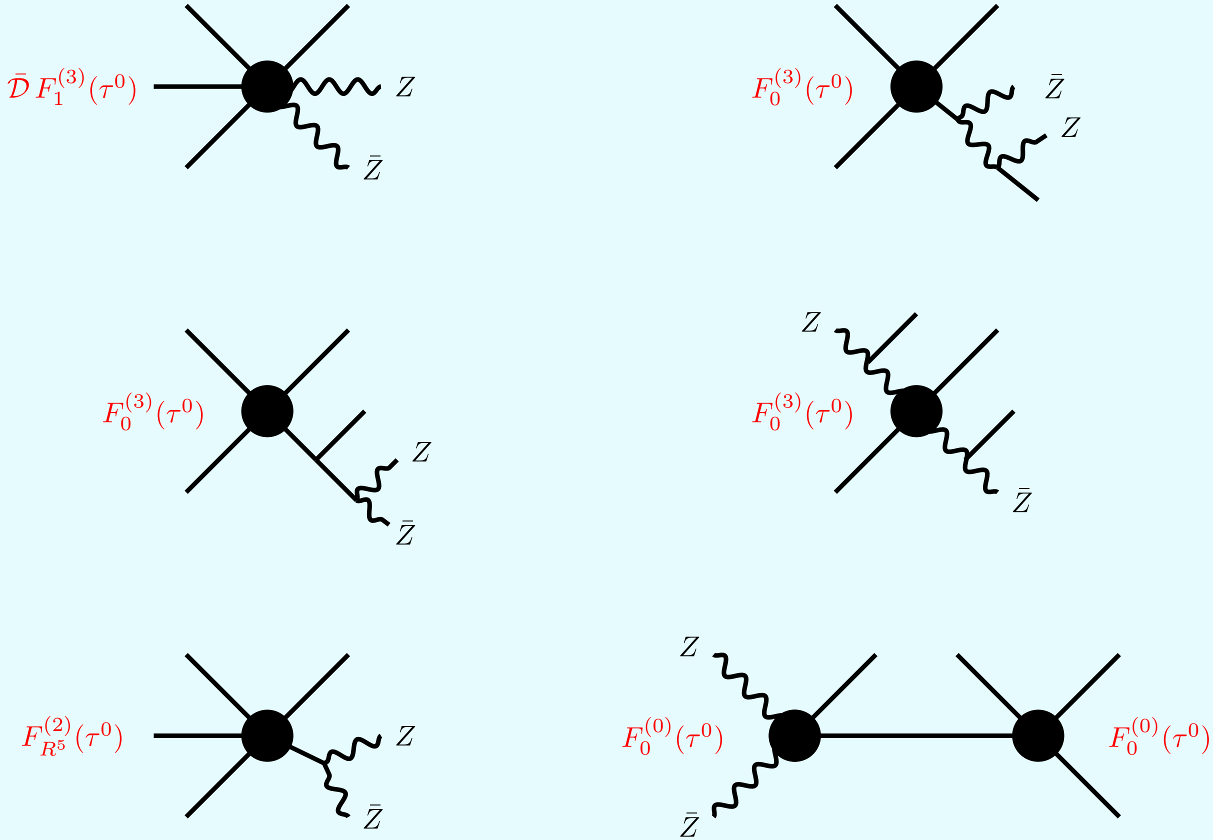
$$\mathcal{O}^{(3)}_{6,1}\big|_{p_i\to 0} \to 0 \qquad \mathcal{O}^{(3)}_{6,2}\big|_{p_i\to 0} \to 0$$

$n>6, \quad p=3$

$$\mathcal{O}^{(3)}_{n,1} = \frac{1}{32}\left((28-3n)\sum_{i<j} s^3_{ij} + 3\sum_{i<j<k} s^3_{ijk}\right) \qquad \mathcal{O}^{(3)}_{n,2} = (n-4)\sum_{i<j} s^3_{ij} - \sum_{i<j<k} s^3_{ijk}$$

$$\mathcal{O}^{(3)}_{n,1}\big|_{p_n\to 0} = \mathcal{O}^{(3)}_{n-1,1} \qquad \mathcal{O}^{(2)}_{n,1}\big|_{p_n\to 0} = \mathcal{O}^{(2)}_{n-1,1}$$

e.g. Amplitudes with a Z and a $\bar{Z}$



Supersymmetry forbids a supersymmetric contact interaction - implies a relation of form:

$$\bar{\mathcal{D}}\,\mathcal{E}_1^{(3)}(\tau^0) + a\,\mathcal{E}_0^{(3)}(\tau^0) + b\,E_0(\tfrac{3}{2},\tau^0)E_0(\tfrac{3}{2},\tau^0) = 0$$

$\mathcal{E}_1(\tau) = 2\,\mathcal{D}\mathcal{E}_0(\tau)$ gives Laplace eqn.

$$\bar{\mathcal{D}}\mathcal{D}\,\mathcal{E}_0^{(3)}(\tau^0) + a\mathcal{E}^{(3)}(\tau^0) + bE_0(\tfrac{3}{2},\tau^0)\,E_0(\tfrac{3}{2},\tau^0) = 0$$

Coefficients fixed by comparison with tree-level amplitudes

Comments

- We have determined non-perturbative behaviour of all "protected" terms in the low-energy expansion of the form $\mathcal{E}^{(p)}_{w,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\})$ - up to overall constants $c^{(p)}_{n,i}$ $(n = 4 + w)$ that are determined from tree (or one-loop) amplitudes and (in principle) by supersymmetry.

- These violate the continuous U(1) R-symmetry in string theory by q= -2w units.
  The $w \neq 0$ cases do not arise in maximal supergravity

- These interactions are related by FIRST-ORDER DIFFERENTIAL EQUATIONS – consequence of SUPERSYMMETRY as is apparent from the amplitude calculations.

- LAPLACE EQUATIONS follow as consequence of first-order equations.
  Leading to the same non-renormalisation conditions as in maximal supergravity.

- Generalisations to compactified theory.

- RECALL Leading D-instanton terms match beautifully with instanton contributions to correlation functions in large-N limit of Yang-Mills – HOLOGRAPHIC MATCHING.

- SUGGESTS A HOLOGRAPHIC ORIGIN FOR THESE AMPLITUDES.

Unconventional limit of large-N SU(N) $\mathcal{N}=4$ supersymmetric Yang-Mills : Basu, MBG, Sethi (2004)

g_{YM} **FIXED AND** $N\to\infty$ **PRESERVES** $SL(2,\mathbb{Z})$ **MONTONEN-OLIVE DUALITY**

Consider derivative of Yang - Mills correlation function $\frac{\partial}{\partial\tau}\langle\, \mathcal{O}_1\, \mathcal{O}_2 \ldots \mathcal{O}_n\rangle$

complex coupling

Leading behaviour $O(N^{1/2})$ of correlation functions (plausibly) satisfies the $O((\alpha')^{-1})$ string theory differential equations discussed in this talk.