BOREL RESUMMATION OF THE CRITICAL EXPONENTS

$\epsilon$-expansion vs conformal bootstrap

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MOTIVATION

\[ S = -\frac{1}{2} (m^2_0 + p^2) \varphi^2_0 - \frac{1}{4!} g_0 \varphi^4_0 \]

Perturbative series of anomalous dimensions and critical exponents computed in the framework of $\epsilon$-expansion ($d = 4 - \epsilon$) are asymptotic series with factorially growing coefficients.

To get reliable estimates of the critical exponents for physical values of $\epsilon$ ($\epsilon = \{1, 2\}$) one need to perform resummation of the expansion. (Usually Borel-like resummation)

Results at $d = 2$ and 3 we can compare with
- high temperature expansion
- Monte-Carlo methods
- conformal bootstrap

Results at $d = 3$ are in a good agreement, while for $d = 2$ exponents differs in worst case up to 15%, which is usually related to slow convergence due to large value of the expansion parameter ($\epsilon = 2$)
Recent conformal bootstrap calculations of critical exponents in diverse dimension allow to perform deep comparison of the Borel resummed $\epsilon$-expansion and conformal bootstrap.

- critical exponents at $d = 3$ are very close to each other
- while at $d = 2$ differs significantly
- authors report that near $d = 2.2$ ($\epsilon = 1.8$) they observe rearrangement of the conformal states in such a way that at $d = 2$ they fit Virasoro representation

\[\eta(\epsilon)\] \[\nu(\epsilon)\] \[\omega(\epsilon)\]

$\eta(\epsilon)$ $\nu(\epsilon)$ $\omega(\epsilon)$

\[\epsilon = 4 - d\]

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we expected that for $d > 2.2$ ($\epsilon < 1.8$) we will have agreement with conformal bootstrap, while after $d = 2.2$ they will differ

our calculations show that difference between exponents occur starting from $d = 3.5$, while for $d > 3.5$ ($\epsilon < 0.5$) they are in perfect agreement

situation with exponent $\omega$ is not completely clear, as for $\phi^4$ model contrary to bootstrap prediction $\omega = 2$ there are alternate predictions:


OVERVIEW

1. Universality classes of $\varphi^4$ model

2. High order asymptotics. Borel Transform

3. Resummation
   3.1 Pade
   3.2 Pade-Borel
   3.3 Conformal mapping

4. Comparison with conformal bootstrap

5. Discussion
UNIVERSALITY CLASSES OF $\varphi^4$ MODEL
\[ S(\varphi) = - \int dx \left( \frac{1}{2} \tau \varphi(x)^2 + \frac{1}{2} (\nabla \varphi(x))^2 + \frac{1}{24} g(\varphi(x)^2)^2 \right) \]

\(O(n)\)-symmetric \(\varphi^4\) model in statistical physics describes second order phase transition in:

- \(n = 0\)
  - self-avoiding walks

- \(n = 1\)
  - liquid-gas system
  - critical mixing point in binary mixtures

- \(n = 2\)
  - planar Heisenberg magnet
  - transition to the superfluid phase of liquid \(^4\text{He}\)

- \(n = 3\)
  - isotropic Heisenberg magnet
ISING UNIVERSALITY CLASS

\[ S = -\frac{1}{2} (m_0^2 + p^2) \varphi_0^2 - \frac{1}{4!} g_0 \varphi_0^4 \]

\( \varphi \) – scalar field \((N = 1)\)

Full set of the critical exponents:

- \( \alpha \) – specific heat \( C \sim \tau^{-\alpha} \)
- \( \beta \) – order parameter \( \varphi \sim \tau^\beta \)
- \( \gamma \) – susceptibility \( \chi \sim \tau^{-\gamma} \)
- \( \delta \) – source field (pressure/external field) at \( T = T_c \): \( P \sim \varphi^\delta \)
- \( \eta \) – two-point correlation function

\[
\langle \phi(0)\phi(r) \rangle = r^{-d+2} \left( \frac{r}{r_0} \right)^{-\eta} F(r/\xi) \quad (F(0) – \text{finite})
\]

- \( \nu \) – correlation length \( \xi = r_0 \tau^{-\nu} \)
- \( \omega \) – correction exponent, e.g.

\[
\langle \phi(0)\phi(r) \rangle = r^{-d+2} \left( \frac{r}{r_0} \right)^{-\eta} \left( F(0) + a_\omega \left( \frac{r_0}{r} \right)^\omega + \ldots \right) \quad (\tau = 0)
\]

Critical exponents satisfy (hyper)scaling relations:

\[ \gamma = \nu(2 - \eta), \quad D\nu = 2 - \alpha, \quad \beta\delta = \beta + \gamma, \quad \alpha + 2\beta + \gamma = 2 \]

only two of the exponents are independent (plus \( \omega \)).

For renormalization group most natural choice is exponents \( \eta \) and \( 1/\nu \):

\[ \eta = 2\gamma^*_\phi, \quad 1/\nu = 2 - \gamma^*_m^2 \]
Progress in perturbative calculations in the framework of renormalization group/\(\epsilon\)-expansion:

- **Critical exponents at 4-loop level:**

- **5-loop order (analytical):**

- **6-loop order (analytical):**

- **7-loop order (analytical):**

- **Primitive\(^2\) graphs**
  - O. Schnetz, *Commun. Number Theory Phys.* 4 (2010), no. 1 1–47 (*analytically, up to 7 loops and almost all 8 loops*)

\(^2\)no subdivergences
RG RESULTS FOR ISING UNIVERSALITY CLASS

Beta function and anomalous dimensions \((D = 4 - 2\varepsilon = 4 - \epsilon)^3:\)

\[
\beta(g) = -\epsilon g + 3g^2 - 5.6667g^3 + 32.5497g^4 - 271.606g^5 + 2848.57g^6 - \\
-34776.1g^7 + 474651g^8 + O(g^9)
\]

\[
\gamma_\phi(g) = 0.083333g^2 - 0.0625g^3 + 0.33854g^4 - 1.9256g^5 + 14.384g^6 - 124.16g^7 + O(g^8)
\]

\[
\gamma_{m^2} = -g + 0.83333g^2 - 3.5g^3 + 19.9563g^4 - 150.756g^5 + 1354.64g^6 - 13759.8g^7 + O(g^8)
\]

Fixed point: \(\beta(g^*) = 0\)

Critical exponents:

\[
\eta = 2\gamma_\phi(g^*) = 0.018518\epsilon^2 + 0.018690\epsilon^3 - 0.0083288\epsilon^4 + 0.025656\epsilon^5 - 0.081273\epsilon^6 + \\
+0.31475\epsilon^7 + O(\epsilon^8)
\]

\[
\frac{1}{\nu} = 2 + \gamma_{m^2}(g^*) = 2 - 0.33333\epsilon - 0.11728\epsilon^2 + 0.12453\epsilon^3 - 0.30685\epsilon^4 + 0.95124\epsilon^5 - \\
-3.5726\epsilon^6 + 15.287\epsilon^7 + O(\epsilon^8)
\]

\[
\omega = \partial_g\beta(g^*) = \epsilon - 0.62963\epsilon^2 + 1.6182\epsilon^3 - 5.2351\epsilon^4 + 20.750\epsilon^5 - 93.111\epsilon^6 + 458.74\epsilon^7 + O(\epsilon^8)
\]

\(^3\)Historically for diagram calculation \(D = 4 - 2\varepsilon\) is used, while for resummation – \(D = 4 - \epsilon\)
HIGH ORDER ASYMPTOTICS. BOREL TRANSFORM
High order asymptotics of perturbative expansions of $\phi^4$ model

$$f(z) = \sum_{n=0}^{\infty} A_n z^n, \quad (z = g, \epsilon)$$

$$A_n = C n!(-a)^n n^{b_0} (1 + O(1/n))$$

Series has zero convergence radius.

Borel transform:

$$\sum A_n z^n \rightarrow B(t) = \sum B_n t^n, \quad B_n = \frac{A_n}{\Gamma(n + b + 1)}$$

Transformed series has convergence radius equal to $1/a$

Inverse Borel transform:

$$f^{\text{resum}}(z) = \int_{0}^{\infty} dt \ e^{-t} t^b B(zt)$$

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4. L.N. Lipatov, *JETP Lett.* 25, 104 (1977); *JETP* 45, 216 (1977);

Socal-Watson Theorem

\[ f(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{N-1} A_n z^n + R_N(z), \quad |R_N(z)| \leq A \sigma^N N!|z|^N \quad (1) \]

If eq. (1) satisfied uniformly in \( N \) and \( z \in C_R \). Then \( B(t) = \sum B_n t^n = \sum A_n / n! t^n \) converges for \( |t| < 1/\sigma \equiv s \) and has analytic continuation to \( S_s \). Satisfying the bound:

\[ |B(t)| \leq K \exp(|t|/R) \]

Furthermore, \( f(z) \) can be represented by the absolutely convergent integral

\[ f(z) = \frac{1}{z} \int_0^\infty e^{-t/z} B(t) dt \]

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RESUMMATION
PADE APPROXIMANTS

\[ f(z) = \sum_{n=0}^{N} A_n z^n \rightarrow \tilde{f}(z) = P_{[L/M]}(z) = \frac{P_L(z)}{P_M(z)}, \quad L + M + 1 = N \]

**Benefits:**
- Very simple and fast resummation procedure
- No need to know high order asymptotics

**Drawbacks:**
- Can contain artificial poles in physical range of the expansion parameter
- Usually only near-to-central \((|L - M| \leq 2)\) approximants give reliable predictions at large values of expansion parameter.
- Series with zero convergence radius

**Error estimation:**
- Drop out all approximants with poles in physical range of expansion parameter
- Drop out all approximants which produce obviously unreliable results
- Consider all survived approximants as independent measurements and compute estimate with:

\[
\langle x \rangle = \frac{x_1 + \ldots + x_n}{n}, \quad \Delta x = t_{0.95, n} \sqrt{\frac{(x_1 - \langle x \rangle)^2 + \ldots + (x_n - \langle x \rangle)^2}{n(n-1)}}.
\]

Where \(t_{p,n}\) is t-distribution with \(p = 0.95\) confidence level.
\[ \eta^{(6)}_{\text{Pade}}(\epsilon = 1) = 0.033(3), \]
\[ \eta^{(7)}_{\text{Pade}}(\epsilon = 1) = 0.037(2), \]
\[ \eta^{(6)}_{\text{CM}}(\epsilon = 1) = 0.03620(60), \]
\[ \eta_{\text{CB}}(\epsilon = 1) = 0.03640(60), \]
\[ \eta^{(6)}_{\text{Pade}}(\epsilon = 2) = 0.20229(8) \]
\[ \eta^{(7)}_{\text{Pade}}(\epsilon = 2) = 0.28(8) \]
\[ \eta^{(6)}_{\text{CM}}(\epsilon = 2) = 0.237(27) \]
\[ \eta_{\text{CB}}(\epsilon = 2) = 0.25 \]
CRITICAL EXPONENTS. (PADE)

\(\eta^{(6)}_{\text{Pade}}(\epsilon = 1) = 0.033(3), \quad \eta^{(6)}_{\text{Pade}}(\epsilon = 2) = 0.20229(8)\)
\(\eta^{(7)}_{\text{Pade}}(\epsilon = 1) = 0.037(2), \quad \eta^{(7)}_{\text{Pade}}(\epsilon = 2) = 0.28(8)\)

\(\eta^{(6)}_{\text{CM}}(\epsilon = 1) = 0.03620(60), \quad \eta^{(6)}_{\text{CM}}(\epsilon = 2) = 0.237(27)\)
\(\eta^{(7)}_{\text{CM}}(\epsilon = 1) = 0.037(2), \quad \eta^{(7)}_{\text{CM}}(\epsilon = 2) = 0.25\)

\(\nu^{(6)}_{\text{Pade}}(\epsilon = 1) = 0.633(4), \quad \nu^{(6)}_{\text{Pade}}(\epsilon = 2) = 0.98(4)\)
\(\nu^{(7)}_{\text{Pade}}(\epsilon = 1) = 0.623(6), \quad \nu^{(7)}_{\text{Pade}}(\epsilon = 2) = 0.92(3)\)

\(\nu^{(6)}_{\text{CM}}(\epsilon = 1) = 0.62920(50), \quad \nu^{(6)}_{\text{CM}}(\epsilon = 2) = 0.952(14)\)
\(\nu^{(7)}_{\text{CM}}(\epsilon = 1) = 0.63005(45), \quad \nu^{(7)}_{\text{CM}}(\epsilon = 2) = 1\)

\(\omega^{(6)}_{\text{Pade}}(\epsilon = 1) = 0.77(7), \quad \omega^{(6)}_{\text{Pade}}(\epsilon = 2) = 1.53164(2)\)
\(\omega^{(7)}_{\text{Pade}}(\epsilon = 1) = 0.83(2), \quad \omega^{(7)}_{\text{Pade}}(\epsilon = 2) = 1.9(4)\)

\(\omega^{(6)}_{\text{CM}}(\epsilon = 1) = 0.820(7), \quad \omega^{(6)}_{\text{CM}}(\epsilon = 2) = 1.71(9)\)
\(\omega^{(7)}_{\text{CM}}(\epsilon = 1) = 0.84(4), \quad \omega^{(7)}_{\text{CM}}(\epsilon = 2) = 2\)

\(\omega_{\text{theor}}(\epsilon = 2) = \{4/3, 1.75, 2\}\)
RESUMMATION WITH BOREL TRANSFORM

Perturbative expansion provides us only with limited number of terms:

\[ f^{(N)}(z) = \sum_{n=0}^{N} A_n z^n \]

Borel transform:

\[ B^{(N)}(t) = \sum_{n=0}^{N} B_n t^n, \quad B_n = \frac{A_n}{\Gamma(n + b + 1)} \]

Inverse Borel transform: \( f^{(N)}_{\text{resum}}(z) = \int_0^\infty dt \ e^{-t} t^b B^{(N)}(zt) \) is trivial.

We need to replace \( B^{(N)}(t) \) by some nontrivial function \( \tilde{B}^{(n)}(t) \), so that:

\[ \tilde{B}^{(N)}(t) = \sum_{n=0}^{N} B_n t^n + O(t^{N+1}) \]

Main problem: there is a plenty variants of realization of \( \tilde{B}^{(N)}(t) \):

- proper choice of \( \tilde{B}^{(N)}(t) \) may significantly increase convergence
- improper – may lead to completely inconsistent results.
PADE-BOREL APPROXIMANTS

\[ f(z) = \sum_{n=0}^{N} A_n z^n \quad \rightarrow \quad B^{(N)}(t) = \sum_{n=0}^{N} \frac{A_n}{\Gamma(n + b + 1)} \quad \rightarrow \quad \tilde{B}^{(N)}(t) = P_{[L/M]}(t) = \frac{P_L(t)}{P_M(t)} \]

Benefits:

- still very simple and fast resummation procedure
- no need to know high order asymptotics
- \( b \) – fitting parameter, proper choice increase convergence
- series for Pade has finite convergence radius

Drawbacks:

- can contain artificial poles on positive axis
- usually only near-to-central (\(|L - M| \leq 2\)) approximants give reliable predictions at large values of expansion parameter.

Error estimation:

- (for each \( b \)) we compute error estimate in the same way as in Pade
- optimal \( b \) is one which minimizes error estimate

this talk \( b = 0 \) only!
CRITICAL EXPONENT $\eta$. (PADE-BOREL)

\[ \eta^{(6)}_{PB}(\epsilon = 1) = 0.034(4), \]
\[ \eta^{(7)}_{PB}(\epsilon = 1) = 0.0363(0) \]
\[ \eta^{(6)}_{CM}(\epsilon = 1) = 0.03620(60), \]
\[ \eta^{(6)}_{CB}(\epsilon = 1) = 0.03640(60), \]
\[ \eta^{(6)}_{PB}(\epsilon = 2) = 0.217(0), \]
\[ \eta^{(7)}_{PB}(\epsilon = 2) = 0.244(0) \]
\[ \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27), \]
\[ \eta^{(6)}_{CB}(\epsilon = 2) = 0.25 \]
CRITICAL EXPONENTS. (PADE-BOREL)

\[\eta^{(6)}_{PB}(\epsilon = 1) = 0.034(4), \quad \eta^{(6)}_{PB}(\epsilon = 2) = 0.217(0)\]
\[\eta^{(7)}_{PB}(\epsilon = 1) = 0.0363(0), \quad \eta^{(7)}_{PB}(\epsilon = 2) = 0.244(0)\]
\[\eta^{(6)}_{CM}(\epsilon = 1) = 0.03620(60), \quad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)\]
\[\eta^{(6)}_{CB}(\epsilon = 1) = 0.03640(60), \quad \eta^{(6)}_{CB}(\epsilon = 2) = 0.25\]

\[\nu^{(6)}_{PB}(\epsilon = 1) = 0.627(3), \quad \nu^{(6)}_{PB}(\epsilon = 2) = 0.904(1)\]
\[\nu^{(7)}_{PB}(\epsilon = 1) = 0.6(1), \quad \nu^{(7)}_{PB}(\epsilon = 2) = 1.082(0)\]
\[\nu^{(6)}_{CM}(\epsilon = 1) = 0.62920(50), \quad \nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14)\]
\[\nu^{(6)}_{CB}(\epsilon = 1) = 0.63005(45), \quad \nu^{(6)}_{CB}(\epsilon = 2) = 1\]

\[\omega^{(6)}_{PB}(\epsilon = 1) = 0.85(13), \quad \omega^{(6)}_{PB}(\epsilon = 2) = 1.567(0)\]
\[\omega^{(7)}_{PB}(\epsilon = 1) = 0.833(0), \quad \omega^{(7)}_{PB}(\epsilon = 2) = 1.872(0)\]
\[\omega^{(6)}_{CM}(\epsilon = 1) = 0.820(7), \quad \omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9)\]
\[\omega^{(6)}_{CB}(\epsilon = 1) = 0.84(4), \quad \omega^{(6)}_{CB}(\epsilon = 2) = 2\]
\[\omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\}\]
Benefits:
- very fast and simple

Drawbacks:
- problems with determination of the “propper” approximants,
- problems with determination of the error estimates,
- low accuracy at large values of the expansion parameter.
CONFORMAL MAPPING

\[ f(z) = \sum_{n=0}^{N} A_n z^n \rightarrow B^{(N)}(t) = \sum_{n=0}^{N} \frac{A_n}{\Gamma(n + b + 1)} \rightarrow \]

\[ \rightarrow \tilde{B}^{(N)}(t) = F^{(N)}(w), \quad w(t) = \frac{\sqrt{1 - at + 1}}{\sqrt{1 + at + 1}} \]

Maps integration domain into \([0, 1)\)

Inverse Borel transformation:

\[ f^{(N)}_{\text{resum}} = \int_0^{\infty} dt e^{-t} t^b F^N(w(t)) \]

if \( b = b_0 + 3/2 \) provides proper high order asymptotics for resummed function

\( \sim n! (-a)^n n^{b_0} \)
\[
\begin{align*}
  f(z) &= \sum_{n=0}^{N} A_n z^n \
  B^{(N)}(t) &= \sum_{n=0}^{N} \frac{A_n}{\Gamma(n+b+1)} \\
  \tilde{B}^{(N)}(t) &= F^{(N)}(w), \quad w(t) = \frac{\sqrt{1-at+1}}{\sqrt{1+at+1}}
\end{align*}
\]

Benefits:
- successive approximations
- incorporation of the high order asymptotics increase convergence
- it is possible to incorporate other properties of the series (e.g. strong coupling asymptotics), also increase convergence

Drawbacks:
- necessary to avoid introducing too much fitting parameters
- too much variants for implementation of the function \( \tilde{B}^{(N)}(t) \)

Error estimation:

scan over fitting parameters and minimize some functional (error estimate), fitting parameters which provide most stable results are considered to be optimal.
CONFORMAL MAPPING

- no fitting parameters (only high order asymptotics)
  bad convergence

- adaptive (K.Wiese):
  high order parameters determined from series coefficients

- free boundary condition (L.Adzhemyan, E.Ivanova):
  enforce series to converge to some value at $\epsilon = 2$ (this values is used as fitting parameter, as well as strong coupling asymptotic)$^7$

- h./sc./b (M.Kompaniets, E.Panzer) optimization over 3 parameters:
  homographic transform, strong coupling asymptotics and $b$.

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$^7$improvement of “boundary condition” method by R.Guida, J.Zinn-Justin: enforce series to coincide with Onsager solution at $\epsilon = 2$
ADAPTIVE CONFORMAL MAPPING

\[ f(z) = \sum_{n=0}^{N} A_n z^n \quad A_n \sim n!(-a)^n n^b \]

constructing ratios:

\[ r_n = \frac{A_n}{A_{n-1}} \frac{1}{n} \left( \frac{n}{n-1} \right)^b = -a + \delta a(n), \quad \delta a(n) = b e^{-cn} \]

(c > 0, b is fitting parameter)

we can extract from the series

- position of the branch cut \((-1/a)\)
- estimate high order coefficients \((A_n)\) up to \(n = 20 \ldots 40\), to enforce series after conformal mapping to share the properties of the original series.

For this new series we use conformal mapping procedure, while varying \(b\) we got uncertainty estimation.
ADAPTIVE CONFORMAL MAPPING: RESULTS

\[ \eta^{(6)}_{Ad}(\epsilon = 1) = 0.03503(5), \quad \eta^{(6)}_{Ad}(\epsilon = 2) = 0.2022(8) \]
\[ \eta^{(6)}_{CM}(\epsilon = 1) = 0.0362(6), \quad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27) \]
\[ \eta^{(6)}_{CB}(\epsilon = 1) = 0.03640(60), \quad \eta^{(6)}_{CB}(\epsilon = 2) = 0.25 \]

\[ \nu^{(6)}_{Ad}(\epsilon = 1) = 0.6288(2), \quad \nu^{(6)}_{Ad}(\epsilon = 2) = 0.936(4) \]
\[ \nu^{(6)}_{CM}(\epsilon = 1) = 0.6292(5), \quad \nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14) \]
\[ \nu^{(6)}_{CB}(\epsilon = 1) = 0.63005(45), \quad \nu^{(6)}_{CB}(\epsilon = 2) = 1 \]

\[ \omega^{(6)}_{Ad}(\epsilon = 1) = 0.831(9), \quad \omega^{(6)}_{Ad}(\epsilon = 2) = 1.81(12) \]
\[ \omega^{(6)}_{CM}(\epsilon = 1) = 0.820(7), \quad \omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9) \]
\[ \omega^{(6)}_{CB}(\epsilon = 1) = 0.84(4), \quad \omega^{(6)}_{CB}(\epsilon = 2) = 2 \]
\[ \omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\} \]

To be fixed: implement more conservative uncertainties
FREE BOUNDARY CONDITION

\[
f(\epsilon) = \sum_{n=0}^{N} A_n \epsilon^n = f_2 + (2 - \epsilon) G(\epsilon),
\]

where \( f_2 = f(2) \) is one of our fitting parameters.\(^8\)

\[
G(\epsilon) = \frac{f(\epsilon) - f_2}{2 - \epsilon} = \sum_{n=0}^{N} G_n \epsilon^n + \mathcal{O}(\epsilon)
\]

We perform resummation of the \( G(\epsilon) \) with conformal mapping

\[
G^{(N)}(\epsilon) = \sum_{n=0}^{N} G_n \epsilon^n \rightarrow B^{(N)}(t) = \sum_{n=0}^{N} \frac{G_n}{\Gamma(n + b + 1)} t^n \rightarrow \\
\rightarrow \tilde{B}^{(N)}(t) = \left( \frac{t}{w(t)} \right)^{\lambda} \sum_{n=1}^{N} \tilde{B}_n w(t)^n
\]

\( \lambda \) is another fitting parameter which governs strong coupling asymptotics.

To determine the optimal fitting parameters we optimize the quantity:

\[
Q(\lambda, f_2) = \sqrt{\left( \partial_\lambda (f^{(N)}_{\lambda, f_2} - f^{(N-1)}_{\lambda, f_2}) \right)^2 + \left( \partial_\lambda f^{(N)}_{\lambda, f_2} \right)^2}
\]

\(^8\) improvement of “boundary condition” method by R.Guida, J.Zinn-Justin: enforce series to coincide with Onsager solution at \( \epsilon = 2 \)
FREE BOUNDARY CONDITION

\[ Q(\lambda, f_2) = \sqrt{\left( \partial_\lambda (f_\lambda^{(N)}) - f_\lambda^{(N-1)}) \right)^2 + \left( \partial_\lambda f_\lambda^{(N)} \right)^2} \]

Optimization criterion constructed intended to select parameters where we have stable values with accuracy increasing from order to order:
FREE BOUNDARY CONDITION

In the fitting parameter space $Q(\lambda, f_2)$ has sharp minimum

\[ \eta_{fbc}(\epsilon = 1) = 0.0355(5) \]
\[ \eta_{fbc}(\epsilon = 1) = 0.035(1) \]
\[ \eta_{fbc}(\epsilon = 2)_{\text{fit}} = 0.21(1) \]
\[ \eta_{fbc}(\epsilon = 2)_{\text{fit}} = 0.21(3) \]

\[ \nu_{fbc}(\epsilon = 1) = 0.629(1) \]
\[ \nu_{fbc}(\epsilon = 1) = 0.6293(4) \]
\[ \nu_{fbc}(\epsilon = 2)_{\text{fit}} = 0.94(4) \]
\[ \nu_{fbc}(\epsilon = 2)_{\text{fit}} = 0.943(9) \]

\[ \omega_{fbc}(\epsilon = 1) = 0.8106(4) \]
\[ \omega_{fbc}(\epsilon = 1) = 0.8125(5) \]
\[ \omega_{fbc}(\epsilon = 2)_{\text{fit}} = 1.560(3) \]
\[ \omega_{fbc}(\epsilon = 2)_{\text{fit}} = 1.575(3) \]
FREE BOUNDARY CONDITION. RESULTS

\[ \eta_{fbc}(\epsilon = 1) = 0.0355(5), \]
\[ \eta_{fbc}(\epsilon = 1) = 0.035(1), \]
\[ \eta_{CM}(\epsilon = 1) = 0.0362(6), \]
\[ \eta_{CB}(\epsilon = 1) = 0.03640(60), \]
\[ \nu_{fbc}(\epsilon = 1) = 0.629(1), \]
\[ \nu_{fbc}(\epsilon = 1) = 0.6293(4), \]
\[ \nu_{CM}(\epsilon = 1) = 0.6292(5), \]
\[ \nu_{CB}(\epsilon = 1) = 0.63005(45), \]
\[ \omega_{fbc}(\epsilon = 1) = 0.8106(5), \]
\[ \omega_{fbc}(\epsilon = 1) = 0.8125(5), \]
\[ \omega_{CM}(\epsilon = 1) = 0.820(7), \]
\[ \omega_{CB}(\epsilon = 1) = 0.84(4), \]
\[ \omega_{fbc}(\epsilon = 2)_{fit} = 0.21(1) \]
\[ \eta_{fbc}(\epsilon = 2)_{fit} = 0.21(3) \]
\[ \eta_{CM}(\epsilon = 2) = 0.237(27) \]
\[ \eta_{CB}(\epsilon = 2) = 0.25 \]
\[ \nu_{fbc}(\epsilon = 2)_{fit} = 0.94(4) \]
\[ \nu_{fbc}(\epsilon = 2)_{fit} = 0.943(9) \]
\[ \nu_{CM}(\epsilon = 2) = 0.952(14) \]
\[ \nu_{CB}(\epsilon = 2) = 1 \]
\[ \nu_{fbc}(\epsilon = 2)_{fit} = 1.560(3) \]
\[ \omega_{fbc}(\epsilon = 2)_{fit} = 1.575(3) \]
\[ \omega_{CM}(\epsilon = 2) = 1.71(9) \]
\[ \omega_{CB}(\epsilon = 2) = 2 \]
\[ \omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\} \]

To be fixed: implement more conservative uncertainties
Homographic transformation $\epsilon = \epsilon'/(1 + q\epsilon')$

$$f(\epsilon) = \sum_{n=0}^{N} A_n \epsilon^n \rightarrow G(\epsilon') = \sum_{n=0}^{N} G_n(\epsilon')^n$$

is intended to soften singularities at large values of $\epsilon$. \(^9\)

$$G^{(N)}(\epsilon') = \sum_{n=0}^{N} G_n(\epsilon')^n \rightarrow B^{(N)}(t) = \sum_{n=0}^{N} \frac{G_n}{\Gamma(n+b+1)} t^n \rightarrow$$

$$\rightarrow \tilde{B}^{(N)}(t) = \left( \frac{t}{w(t)} \right)^\lambda \sum_{n=1}^{N} \tilde{B}_n w(t)^n$$

We perform optimization over 3 parameters: $q$, $\lambda$ and $b$ minimizing error estimate defined as \(^{10}\)

$$E^f_N(b, \lambda, q) \equiv \max\{|\tilde{f}^b,\lambda,q_N - \tilde{f}^b,\lambda,q_{N-1}|, |\tilde{f}^b,\lambda,q_N - \tilde{f}^b,\lambda,q_{N-2}|\}$$

$$+ \max\{\mathrm{Var}_b (\tilde{f}^b,\lambda,q_N), \mathrm{Var}_b (\tilde{f}^b,\lambda,q_{N-1})\}$$

$$+ \mathrm{Var}_\lambda (\tilde{f}^b,\lambda,q_N) + \mathrm{Var}_q (\tilde{f}^b,\lambda,q_N).$$

\(^9\)J.Zinn-Justin

$E^f_N(b, \lambda, q) \equiv \max\{|\tilde{f}^b,\lambda,q_N - \tilde{f}^b,\lambda,q_{N-1}|, |\tilde{f}^b,\lambda,q_N - \tilde{f}^b,\lambda,q_{N-2}|\}$

$$+ \max\{\text{Var}_b (\tilde{f}^b,\lambda,q_N), \text{Var}_b (\tilde{f}^b,\lambda,q_{N-1})\}$$

$$+ \text{Var}_\lambda (\tilde{f}^b,\lambda,q_N) + \text{Var}_q (\tilde{f}^b,\lambda,q_N).$$

Such a complicated error estimation is intended to not underestimate uncertainties of the resummation method and tries to take into account almost all factors.

$\eta^{(6)}_{CM}(\epsilon = 1) = 0.0362(6), \quad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)$

$\eta_{CB}(\epsilon = 1) = 0.03640(60), \quad \eta_{CB}(\epsilon = 2) = 0.25$

$\nu^{(6)}_{CM}(\epsilon = 1) = 0.6292(5), \quad \nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14)$

$\nu_{CB}(\epsilon = 1) = 0.63005(45), \quad \nu_{CB}(\epsilon = 2) = 1$

$\omega^{(6)}_{CM}(\epsilon = 1) = 0.820(7), \quad \omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9)$

$\omega_{CB}(\epsilon = 1) = 0.84(4), \quad \omega_{CB}(\epsilon = 2) = 2$

$\omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\}$
COMPARISON WITH CONFORMAL BOOTSTRAP
\[ \eta, \Delta_\sigma = \frac{D}{2} - 1 + \frac{\eta}{2} \]

\[ \eta(\epsilon) - \text{fit} \]

\[ \eta_{\text{Pade}}(\epsilon = 2) = 0.20229(8) \quad \eta_{\text{Pade}}(\epsilon = 2) = 0.28(8) \quad \eta_{\text{PB}}(\epsilon = 2) = 0.217(0) \]

\[ \eta_{\text{PB}}(\epsilon = 2) = 0.244(0) \quad \eta_{\text{Ad}}(\epsilon = 2) = 0.2022(8) \quad \eta_{\text{fbc}}(\epsilon = 2) = 0.21(3) \]

\[ \eta_{\text{CM}}(\epsilon = 2) = 0.237(27) \quad \eta_{\text{CB}}(\epsilon = 2) = 0.25 \]
\[ \nu, \Delta \epsilon = D - \frac{1}{\nu} \]

\[ \nu(\epsilon) - \text{fit} \]

\[ \nu_{Pade}(\epsilon = 2) = 0.98(4) \quad \nu_{Pade}(\epsilon = 2) = 0.92(3) \quad \nu_{PB}(\epsilon = 2) = 0.904(1) \]

\[ \nu_{PB}(\epsilon = 2) = 1.082(0) \quad \nu_{Ad}(\epsilon = 2) = 0.936(4) \quad \nu_{fbc}(\epsilon = 2) = 0.943(9) \]

\[ \nu_{CM}(\epsilon = 2) = 0.952(14) \quad \nu_{CB}(\epsilon = 2) = 1 \]
\[ \Delta_{\epsilon'} = D + \omega \]

\[ \omega_{Pade}(\epsilon = 2) = 1.53164(2) \quad \omega_{Pade}(\epsilon = 2) = 1.9(4) \quad \omega_{PB}(\epsilon = 2) = 1.567(0) \]

\[ \omega_{PB}(\epsilon = 2) = 1.872(0) \quad \omega_{Ad}(\epsilon = 2) = 1.81(12) \quad \omega_{fbc}(\epsilon = 2) = 1.585(3) \]

\[ \omega_{CM}(\epsilon = 2) = 1.71(9) \quad \omega_{CB}(\epsilon = 2) = 2 \quad \omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\} \]
DISCUSSION
DISCUSSION (1/3)

What we got?

1. We didn’t observe any specific behavior of the $\epsilon$-expansion near $d = 2.2$ ($\epsilon = 1.8$) as it was expected from conformal bootstrap study.
2. We observe significant difference in exponents between conformal bootstrap and $\epsilon$-expansion starting from $d = 3.5$ ($\epsilon > 0.5$).
3. Different resummation methods implemented on top of $\epsilon$-expansion provide consistent results which are different from conformal bootstrap.
4. Situation with exponent $\omega$ is not completely clear, as for $\phi^4$ model contrary to bootstrap prediction $\omega = 2$ there are alternate predictions $\omega = 4/3$ and $\omega = 1.75$. 

![Graph showing $\omega(\epsilon)$ fit]

- Conf.bootstrap
- 6 loops
DISCUSSION (2/3)

What may cause such deviations? And what to do?

1. Usually deviations from Onsager solution are argued to slow convergence due to the large value of the expansion parameter ($\epsilon = 2$). As we see deviations starts at $\epsilon \sim 0.75$ which is believed to be small enough.

2. It does not look like problem of the implementation of the resummation algorithm as different methods provide consistent results.

3. Defect of $\epsilon$-expansion?
   3.1 Borel summability does not proven for $d = 4$ (non-summability also)
   3.2 Even if we believe in Borel summability, Socal-Watson theorem guarantied analiticity of the resummed function only inside $C_R$. But outside $C_R$ it may contain singularities which may to prevent to extend results to the physical values of $\epsilon$.

   3.3 We can implement $g$-summation:
   $\epsilon$-summ. \[ \beta(g^*) = 0 \rightarrow g^* = \sum g_n \epsilon^n \rightarrow \eta = \text{resum}_\epsilon (2\gamma\phi(\sum g_n \epsilon^n)) \]

   $g$-summ. \[ \text{resum}_g(\beta)(g^*) = 0 \rightarrow g^* = \text{const} \rightarrow \eta = \text{resum}_g(2\gamma\phi(g))|_{g=g^*} \]

3.4 Renormalization group in fixed space dimensions (noninteger also)
3.5 Convergent expansions:
Shift expansion point of the continual integral from Gaussian one to make expansion convergent.

4. Conformal bootstrap:
4.1 Investigate origin and properties of the conformal states rearrangement observed at $d = 2.2$. Might be it starts much earlier?
4.2 Investigate in details area $4 > d > 3.25$ ($\epsilon < 0.75$) where $\epsilon$-expansion is expected to work properly.
Thank You!