BOREL RESUMMATION OF THE CRITICAL EXPONENTS

ϵ-expansion vs confromal bootstrap

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MOTIVATION

$$S = -\frac{1}{2}(m_0^2 + p^2)\varphi_0^2 - \frac{1}{4!}g_0\varphi_0^4$$

Perturbative series of anomalous dimensions and critical exponents computed in the framework of ϵ-expansion ($d = 4 - \epsilon$) are **asymptotic series with factorially growing coefficients**.

To get reliable estimates of the critical exponents for physical values of ϵ ($\epsilon = \{1, 2\}$) one need to perform **resummation of the expansion**. (Usually Borel-like resummation)

Results at $d = 2$ and 3 we can compare with

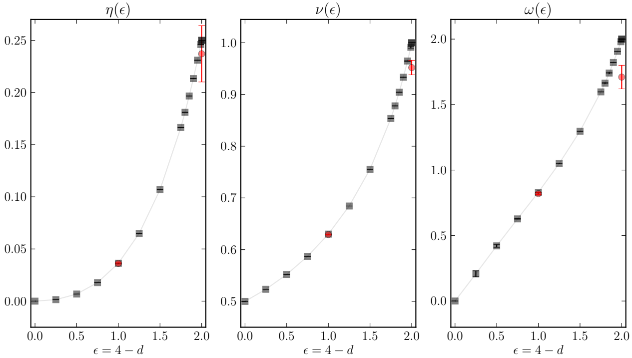
- high temperature expansion
- Monte-Carlo methods
- conformal bootstrap

Results at $d = 3$ are in a good agreement, while for $d = 2$ exponents differs in worst case up to 15%, which is usually **related to slow convergence due to large value of the expansion parameter** ($\epsilon = 2$)

MOTIVATION

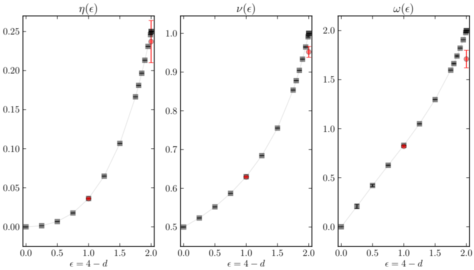
Recent conformal bootstrap calculations of critical exponents in diverse dimension [1] allow to perform deep comparison of the Borel resummed ϵ-expansion and conformal bootstrap.

- critical exponents at $d = 3$ are very close to each other
- while at $d = 2$ differs significantly
- authors report that near $d = 2.2$ ($\epsilon = 1.8$) they observe **rearrangement of the conformal states** in such a way that at $d = 2$ they fit Virasoro representation



[1] A. Cappelli, L. Maffi, S. Okuda, Critical ising model in varying dimension by conformal bootstrap, JHEP 2019 (1) (2019) 161.

MOTIVATION



- we expected that for $d > 2.2$ ($\epsilon < 1.8$) we will have agreement with conformal bootstrap, while after $d = 2.2$ they will differ
- our calculations show that difference between exponents occur starting from $d = 3.5$, while for $d > 3.5$ ($\epsilon < 0.5$) they are in perfect agreement
- situation with exponent ω is not completely clear, as for ϕ^4 model contrary to bootstrap prediction $\omega = 2$ there are alternate predictions:
  - $\omega = 4/3$ by B. Nienhuis, JPA 15 (1) (1982) 199.
    M. Barma, M. E. Fisher, PRL 53 (20) (1984) 1935.
  - $\omega = 1.75$ by P. Calabrese, M. Caselle, A. Celi, A. Pelissetto, and E.Vicari, J. Phys. A 33, 8155 (2000).

OVERVIEW

UNIVERSALITY CLASSES OF φ^4 MODEL

φ^4 MODEL

$$S(\varphi) = -\int d\mathbf{x} \left(\frac{1}{2}\tau\varphi(\mathbf{x})^2 + \frac{1}{2}(\overrightarrow{\nabla}\varphi(\mathbf{x}))^2 + \frac{1}{24}g(\varphi(\mathbf{x})^2)^2 \right)$$

$O(n)$-symmetric φ^4 model in statistical physics describes second order phase transition in:

- $n = 0$
  - self-avoiding walks
- **n=1**
  - liquid-gas system
  - critical mixing point in binary mixtures
- $n = 2$
  - planar Heisenberg magnet
  - transition to the superfluid phase of liquid ^4He
- $n = 3$
  - isotropic Heisenberg magnet

ISING UNIVERSALITY CLASS

$$S = -\frac{1}{2}(m_0^2 + p^2)\varphi_0^2 - \frac{1}{4!} g_0 \varphi_0^4$$

φ – scalar field ($N = 1$)

Full set of the critical exponents:

- α – specific heat $C \sim \tau^{-\alpha}$
- β – order parameter $\varphi \sim \tau^{\beta}$
- γ – susceptibility $\chi \sim \tau^{-\gamma}$
- δ – source field (pressure/external field) at $T = T_c$: $P \sim \varphi^{\delta}$
- η – two-point correlation function

$$\langle \phi(0)\phi(r) \rangle = r^{-d+2} \left(\frac{r}{r_0}\right)^{-\eta} F(r/\xi) \qquad (F(0) \text{– finite})$$

- ν – correlation length $\xi = r_0 \tau^{-\nu}$
- ω – correction exponent, e.g.

$$\langle \phi(0)\phi(r) \rangle = r^{-d+2} \left(\frac{r}{r_0}\right)^{-\eta} \left(F(0) + a_\omega \left(\frac{r_0}{r}\right)^{\omega} + \dots\right) \qquad (\tau = 0)$$

Critical exponents satisfy (hyper)scaling relations:

$$\gamma = \nu(2 - \eta), \quad D\nu = 2 - \alpha, \quad \beta\delta = \beta + \gamma, \quad \alpha + 2\beta + \gamma = 2$$

only two of the exponents are independent (plus ω).
For renormalization group most natural choice is exponents η and $1/\nu$:

$$\eta = 2\gamma_\phi^*, \qquad 1/\nu = 2 - \gamma_{m^2}^*$$

Progress in perturbative calculations in the framework of renormalization group/ϵ-expansion:

- critical exponents at 4-loop level:
  - E. Brezin, J.C. LeGuillou and J. Zinn-Justin, *Phys. Rev.*, D9 (1974) 1121.
  - D.I. Kazakov, O.V. Tarasov and A.A. Vladimirov, *Zh. Eksp. Teor. Fiz.*, 77 (1979) 1035.
- 5-loop order (analytical)
  - K.G. Chetyrkin, A.L. Kataev, F.V. Tkachev 1981 *Phys.Lett.* B 99 147; B 101 457(E)
  - K.G. Chetyrkin , S.G. Gorishny, S.A. Larin and F.V. Tkachov, 1983 *Phys. Lett.* B 132 351
  - D.I. Kazakov 1983 *Phys. Lett.* B 133 406; 1984 *Theor.Math.Phys.* 58 223-230
  - H. Kleinert, J. Neu, V. Shulte-Frohlinde, K.G. Chetyrkin, S.A. Larin 1991 *Phys.Lett.* B 272 39; Erratum 1993 B 319, 545
- 6-loop order (analytical)
  - Batkovich, D.V., Chetyrkin, K.G., Kompaniets, M.V. (2016) *Nuclear Physics B*, 906, pp. 147-167.
  - Kompaniets M., Panzer E. PoS(LL2016)038 (Loops and Legs Proceedings), arXiv:1606.09210
  - Kompaniets M., Panzer E. Phys. Rev. D 96 (2017) 036016, arXiv:1705.06483
- 7-loop order (analytical)
  - Oliver Schnetz Phys. Rev. D 97, (2018) 085018, arXiv:1606.08598v2
- primitive[2] graphs
  - D. J. Broadhurst and D. Kreimer, *Int. J. Mod. Phys. C* 6 (Aug., 1995) 519–524 (numerically, up to 7 loops)
  - O. Schnetz, *Commun. Number Theory Phys.* 4 (2010), no. 1 1–47 (analytically, up to 7 loops and almost all 8 loops)
  - Panzer, E., and Schnetz, O. (2016). The Galois coaction on ϕ^4 periods. arXiv preprint arXiv:1603.04289. (up to 11 loops)

[2] no subdivergences

Beta function and anomalous dimensions ($D = 4 - 2\varepsilon = 4 - \epsilon$)[3]:

$$\beta(g) = -\epsilon g + 3g^2 - 5.6667g^3 + 32.5497g^4 - 271.606g^5 + 2848.57g^6 - \\ -34776.1g^7 + 474651g^8 + \mathcal{O}(g^9)$$

$$\gamma_\phi(g) = 0.083333g^2 - 0.0625g^3 + 0.33854g^4 - 1.9256g^5 + 14.384g^6 - 124.16g^7 + \mathcal{O}(g^8)$$

$$\gamma_{m^2} = -g + 0.83333g^2 - 3.5g^3 + 19.9563g^4 - 150.756g^5 + 1354.64g^6 - 13759.8g^7 + \mathcal{O}(g^8)$$

Fixed point: $\beta(g^*) = 0$

Critical exponents:

$$\eta = 2\gamma_\phi(g^*) = 0.018518\epsilon^2 + 0.018690\epsilon^3 - 0.0083288\epsilon^4 + 0.025656\epsilon^5 - 0.081273\epsilon^6 + \\ +0.31475\epsilon^7 + \mathcal{O}(\epsilon^8)$$

$$1/\nu = 2 + \gamma_{m^2}(g^*) = 2 - 0.33333\epsilon - 0.11728\epsilon^2 + 0.12453\epsilon^3 - 0.30685\epsilon^4 + 0.95124\epsilon^5 - \\ -3.5726\epsilon^6 + 15.287\epsilon^7 + \mathcal{O}(\epsilon^8)$$

$$\omega = \partial_g \beta(g^*) = \epsilon - 0.62963\epsilon^2 + 1.6182\epsilon^3 - 5.2351\epsilon^4 + 20.750\epsilon^5 - 93.111\epsilon^6 + 458.74\epsilon^7 + \mathcal{O}(\varepsilon^8)$$

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[3] Historically for diagram calculation D=$4 - 2\varepsilon$ is used, while for resummation – $D = 4 - \epsilon$

HIGH ORDER ASYMPTOTICS. BOREL TRANSFORM

HIGH ORDER ASYMPTOTICS, BOREL TRANSOFRM

High order asymptotics of perturbative expansions of ϕ^4 model[4] [5]

$$f(z) = \sum_{n=0}^{\infty} A_n z^n, \qquad (z = g, \epsilon)$$

$$A_n = C\ n!(-a)^n n^{b_0}(1 + \mathcal{O}(1/n))$$

Series has zero convergence radius.

Borel transform:

$$\sum A_n z^n \to B(t) = \sum B_n t^n, \quad B_n = \frac{A_n}{\Gamma(n+b+1)}$$

Transformed series has convergence radius equal to $1/a$

Inverse Borel transform:

$$f^{resum}(z) = \int_0^{\infty} dt\ e^{-t} t^b\ B(zt)$$

[4] L.N. Lipatov, *JETP Lett.25, 104 (1977); JETP 45, 216 (1977)*;
E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, *Phys. Rev. D 15, 1544 (1977)*

[5] most recent, detailed analysis: Alan J McKane J. *Phys. A 52, 055401 (2019)*

BOREL SUMMABILITY

Socal-Watson Theorem[6]

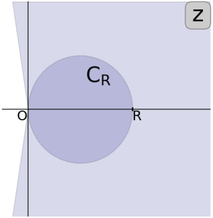
$$f(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{N-1} A_n z^n + R_N(z), \qquad |R_N(z)| \le A\,\sigma^N N! |z|^N \tag{1}$$

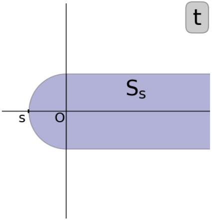
If eq. (1) satisfied uniformaly in N and $z \in C_R$. Then $B(t) = \sum B_n t^n = \sum A_n/n! t^n$ converges for $|t| < 1/\sigma \equiv s$ and has analytic continuation to S_s. Satisfying the bound:

$$|B(t)| \le Kexp(|t|/R)$$

Futhermore, f(z) can be represented by the absolutely convergent integral

$$f(z) = \frac{1}{z} \int_0^{\infty} e^{-t/z} B(t) dt$$





[6] G.N. Watson, *Phil. Trans. A* ***211***, *279 (1911)*;
A.Socal, *J. Math. Phys.* ***21***, *261 (1980)*;

RESUMMATION

PADE APPROXIMANTS

$$f(z) = \sum_{n=0}^{N} A_n z^n \quad \rightarrow \quad \tilde{f}(z) = P_{[L/M]}(z) = \frac{P_L(z)}{P_M(z)}, \quad L + M + 1 = N$$

Benefits:

- very simple and fast resummation procedure
- no need to know high order asymptotics

Drawbacks:

- can contain artificial poles in physical range of the expansion parameter
- usually only near-to-central ($|L - M| \leq 2$) approximants give reliable predictions at large values of expansion parameter.
- series with zero convergence radius

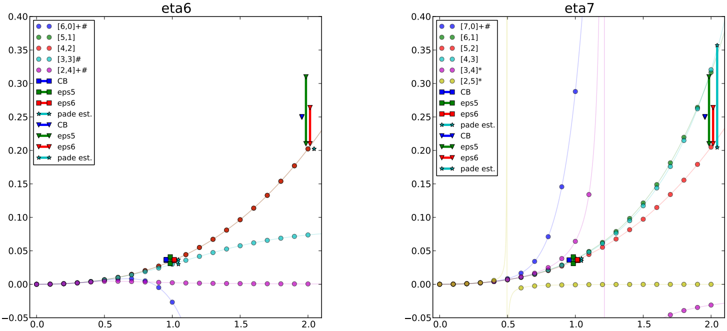
Error estimation:

- drop out all approximants with poles in physical range of expansion parameter
- drop out all approximants which produce obviously unreliable results
- consider all survived approximants as independent measurements and compute estimate with:

$$\langle x \rangle = \frac{x_1 + \ldots + x_n}{n}, \quad \Delta x = t_{0.95,\, n} \sqrt{\frac{(x_1 - \langle x \rangle)^2 + \ldots + (x_n - \langle x \rangle)^2}{n(n-1)}}.$$

where $t_{p,n}$ is t-distribution with p $= 0.95$ confidence level

CRITICAL EXPONENT η. (PADE)



$$\eta^{(6)}_{Pade}(\epsilon = 1) = 0.033(3), \qquad \eta^{(6)}_{Pade}(\epsilon = 2) = 0.20229(8)$$

$$\eta^{(7)}_{Pade}(\epsilon = 1) = 0.037(2), \qquad \eta^{(7)}_{Pade}(\epsilon = 2) = 0.28(8)$$

$$\eta^{(6)}_{CM}(\epsilon = 1) = 0.03620(60), \qquad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)$$

$$\eta_{CB}(\epsilon = 1) = 0.03640(60), \qquad \eta_{CB}(\epsilon = 2) = 0.25$$

CRITICAL EXPONENTS. (PADE)

$$\eta^{(6)}_{Pade}(\epsilon = 1) = 0.033(3), \qquad \eta^{(6)}_{Pade}(\epsilon = 2) = 0.20229(8)$$
$$\eta^{(7)}_{Pade}(\epsilon = 1) = 0.037(2), \qquad \eta^{(7)}_{Pade}(\epsilon = 2) = 0.28(8)$$
$$\eta^{(6)}_{CM}(\epsilon = 1) = 0.03620(60), \qquad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)$$
$$\eta_{CB}(\epsilon = 1) = 0.03640(60), \qquad \eta_{CB}(\epsilon = 2) = 0.25$$

$$\nu^{(6)}_{Pade}(\epsilon = 1) = 0.633(4), \qquad \nu^{(6)}_{Pade}(\epsilon = 2) = 0.98(4)$$
$$\nu^{(7)}_{Pade}(\epsilon = 1) = 0.623(6), \qquad \nu^{(7)}_{Pade}(\epsilon = 2) = 0.92(3)$$
$$\nu^{(6)}_{CM}(\epsilon = 1) = 0.62920(50), \qquad \nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14)$$
$$\nu_{CB}(\epsilon = 1) = 0.63005(45), \qquad \nu_{CB}(\epsilon = 2) = 1$$

$$\omega^{(6)}_{Pade}(\epsilon = 1) = 0.77(7), \qquad \omega^{(6)}_{Pade}(\epsilon = 2) = 1.53164(2)$$
$$\omega^{(7)}_{Pade}(\epsilon = 1) = 0.83(2), \qquad \omega^{(7)}_{Pade}(\epsilon = 2) = 1.9(4)$$
$$\omega^{(6)}_{CM}(\epsilon = 1) = 0.820(7), \qquad \omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9)$$
$$\omega_{CB}(\epsilon = 1) = 0.84(4), \qquad \omega_{CB}(\epsilon = 2) = 2$$
$$\omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\}$$

RESUMMATION WITH BOREL TRANSFORM

Perturbative expansion provides us only with limited number of terms:

$$f^{(N)}(z) = \sum_{n=0}^{N} A_n z^n$$

Borel transform:

$$B^{(N)}(t) = \sum_{n=0}^{N} B_n t^n, \quad B_n = \frac{A_n}{\Gamma(n+b+1)}$$

Inverse Borel transform: $f^{(N)}_{resum}(z) = \int\limits_0^\infty dt\; e^{-t} t^b B^{(N)}(zt)$ is trivial.

We need to replace $B^{(N)}(t)$ by some nontrivial function $\widetilde{B}^{(n)}(t)$, so that:

$$\widetilde{B}^{(N)}(t) = \sum_{n=0}^{N} B_n t^n + \mathcal{O}(t^{N+1})$$

Main problem: there is a plenty variants of relization of $\widetilde{B}^{(N)}(t)$:

- proper choice of $\widetilde{B}^{(N)}(t)$ may significantly increase convergence
- imporper – may lead to completely inconsistent results.

PADE-BOREL APPROXIMANTS

$$f(z) = \sum_{n=0}^{N} A_n z^n \quad \rightarrow \quad B^{(N)}(t) = \sum_{n=0}^{N} \frac{A_n}{\Gamma(n+b+1)} \quad \rightarrow \quad \widetilde{B}^{(N)}(t) = P_{[L/M]}(t) = \frac{P_L(t)}{P_M(t)}$$

Benefits:

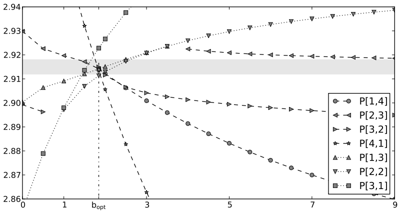
- still very simple and fast resummation procedure
- no need to know high order asymptotics
- b – fitting parameter, proper choice increase convergence
- series for Pade has finite convergence radius

Drawbacks:

- can contain artificial poles on positive axis
- usually only near-to-central ($|L-M| \leq 2$) approximants give reliable predictions at large values of expansion parameter.

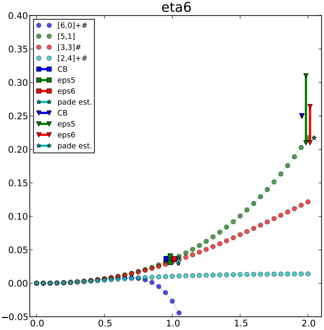
Error estimation:

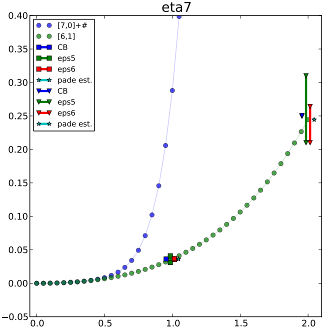
- (for each b) we compute error estimate in the same way as in Pade
- optimal b is one which minimizes error estimate



this talk $b = 0$ only!

CRITICAL EXPONENT η. (PADE-BOREL)





$$\eta_{PB}^{(6)}(\epsilon = 1) = 0.034(4), \qquad \eta_{PB}^{(6)}(\epsilon = 2) = 0.217(0)$$

$$\eta_{PB}^{(7)}(\epsilon = 1) = 0.0363(0) \qquad \eta_{PB}^{(7)}(\epsilon = 2) = 0.244(0)$$

$$\eta_{CM}^{(6)}(\epsilon = 1) = 0.03620(60), \qquad \eta_{CM}^{(6)}(\epsilon = 2) = 0.237(27)$$

$$\eta_{CB}(\epsilon = 1) = 0.03640(60), \qquad \eta_{CB}(\epsilon = 2) = 0.25$$

CRITICAL EXPONENTS. (PADE-BOREL)

$$\eta^{(6)}_{PB}(\epsilon=1)=0.034(4), \qquad \eta^{(6)}_{PB}(\epsilon=2)=0.217(0)$$
$$\eta^{(7)}_{PB}(\epsilon=1)=0.0363(0) \qquad \eta^{(7)}_{PB}(\epsilon=2)=0.244(0)$$
$$\eta^{(6)}_{CM}(\epsilon=1)=0.03620(60), \qquad \eta^{(6)}_{CM}(\epsilon=2)=0.237(27)$$
$$\eta_{CB}(\epsilon=1)=0.03640(60), \qquad \eta_{CB}(\epsilon=2)=0.25$$

$$\nu^{(6)}_{PB}(\epsilon=1)=0.627(3), \qquad \nu^{(6)}_{PB}(\epsilon=2)=0.904(1)$$
$$\nu^{(7)}_{PB}(\epsilon=1)=0.6(1), \qquad \nu^{(7)}_{PB}(\epsilon=2)=1.082(0)$$
$$\nu^{(6)}_{CM}(\epsilon=1)=0.62920(50), \qquad \nu^{(6)}_{CM}(\epsilon=2)=0.952(14)$$
$$\nu_{CB}(\epsilon=1)=0.63005(45), \qquad \nu_{CB}(\epsilon=2)=1$$

$$\omega^{(6)}_{PB}(\epsilon=1)=0.85(13), \qquad \omega^{(6)}_{PB}(\epsilon=2)=1.567(0)$$
$$\omega^{(7)}_{PB}(\epsilon=1)=0.833(0), \qquad \omega^{(7)}_{PB}(\epsilon=2)=1.872(0)$$
$$\omega^{(6)}_{CM}(\epsilon=1)=0.820(7), \qquad \omega^{(6)}_{CM}(\epsilon=2)=1.71(9)$$
$$\omega_{CB}(\epsilon=1)=0.84(4), \qquad \omega_{CB}(\epsilon=2)=2$$
$$\omega_{theor}(\epsilon=2)=\{4/3, 1.75, 2\}$$

PADE / PADE-BOREL APPROXIMANTS

Benefits:

- very fast and simple

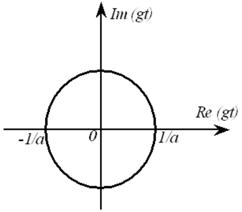
Drawbacks:

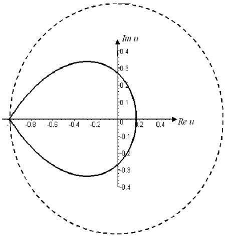
- problems with determination of the "propper" approximants,
- problems with determination of the error estimates,
- low accuracy at large values of the expansion parameter.

CONFORMAL MAPPING

$$f(z) = \sum_{n=0}^{N} A_n z^n \;\;\rightarrow\;\; B^{(N)}(t) = \sum_{n=0}^{N} \frac{A_n}{\Gamma(n+b+1)} \;\;\rightarrow$$

$$\rightarrow\;\; \widetilde{B}^{(N)}(t) = F^{(N)}(w), \;\; w(t) = \frac{\sqrt{1-at}+1}{\sqrt{1+at}+1}$$





Maps integration domain into $[0,1)$

Inverse Borel transformation:

$$f^{(N)}_{resum} = \int_0^\infty dt e^{-t} t^b F^N(w(t))$$

if $b = b_0 + 3/2$ provides proper high order asymptotics for resummed function ($\sim n!(-a)^n n^{b_0}$)

$$f(z) = \sum_{n=0}^{N} A_n z^n \;\to\; B^{(N)}(t) = \sum_{n=0}^{N} \frac{A_n}{\Gamma(n+b+1)} \;\to$$

$$\to\; \widetilde{B}^{(N)}(t) = F^{(N)}(w), \quad w(t) = \frac{\sqrt{1-at}+1}{\sqrt{1+at}+1}$$

Benefits:

- successive approximations
- incorporation of the high order asymptotics increase convergence
- it is possible to incorporate other properties of the series (e.g. strong coupling asymptotics), also increase convergence

Drawbacks:

- necessary to avoid introducing too much fitting parameters
- too much variants for implementation of the function $\widetilde{B}^{(N)}(t)$

Error estimation:

scan over fitting parameters and minimize some functional (error estimate), fitting parameters which provide most stable results are considered to be optimal.

CONFORMAL MAPPING

- no fitting parameters (only high order asymptotics)
  bad convergence
- adaptive (K.Wiese):
  high order parameters determined from series coefficients
- free boundary condition (L.Adzhemyan, E.Ivanova):
  enforce series to converge to some value at $\epsilon = 2$ (this values is used as fitting parameter, as well as strong coupling asymptotic)[7]
- h./sc./b (M.Kompaniets, E.Panzer) optimization over 3 parameters:
  homographic transform, strong coupling asymptotics and *b*.

[7] improvement of "boundary condition" method by R.Guida, J.Zinn-Justin: enforce series to coincide with Onsager solution at $\epsilon = 2$

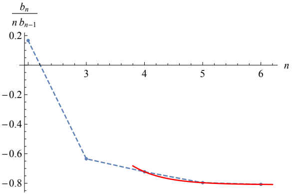
$$f(z) = \sum_{n=0}^{N} A_n z^n \qquad A_n \sim n!(-a)^n n^b$$

constructing ratios:

$$r_n = \frac{A_n}{A_{n-1}} \frac{1}{n} \left(\frac{n}{n-1} \right)^b = -a + \delta a(n), \quad \delta a(n) = b e^{-cn}$$

($c > 0$, b is fitting parameter)
we can extract from the series

- position of the branch cut ($-1/a$)
- estimate high order coefficients (A_n) up to $n = 20 \ldots 40$, to enforce series after conformal mapping to share the properties of the original series.



For this new series we use conformal mapping procedure, while varying b we got uncertainty estimation.

ADAPTIVE CONFORMAL MAPPING. RESULTS

$$\eta^{(6)}_{Ad}(\epsilon = 1) = 0.03503(5), \qquad \eta^{(6)}_{Ad}(\epsilon = 2) = 0.2022(8)$$

$$\eta^{(6)}_{CM}(\epsilon = 1) = 0.0362(6), \qquad \eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)$$

$$\eta_{CB}(\epsilon = 1) = 0.03640(60), \qquad \eta_{CB}(\epsilon = 2) = 0.25$$

$$\nu^{(6)}_{Ad}(\epsilon = 1) = 0.6288(2), \qquad \nu^{(6)}_{Ad}(\epsilon = 2) = 0.936(4)$$

$$\nu^{(6)}_{CM}(\epsilon = 1) = 0.6292(5), \qquad \nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14)$$

$$\nu_{CB}(\epsilon = 1) = 0.63005(45), \qquad \nu_{CB}(\epsilon = 2) = 1$$

$$\omega^{(6)}_{Ad}(\epsilon = 1) = 0.831(9), \qquad \omega^{(6)}_{Ad}(\epsilon = 2) = 1.81(12)$$

$$\omega^{(6)}_{CM}(\epsilon = 1) = 0.820(7), \qquad \omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9)$$

$$\omega_{CB}(\epsilon = 1) = 0.84(4), \qquad \omega_{CB}(\epsilon = 2) = 2$$

$$\omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\}$$

To be fixed: implement more conservative uncertainties

FREE BOUNDARY CONDITION

$$f(\epsilon) = \sum_{n=0}^{N} A_n \epsilon^n = f_2 + (2-\epsilon) G(\varepsilon)\,,$$

where $f_2 = f(2)$ is one of our fitting parameters.[8]

$$G(\epsilon) = \frac{f(\varepsilon) - f_2}{2-\epsilon} = \sum_{n=0}^{N} G_n \varepsilon^n + \mathcal{O}(\epsilon)$$

We perform resummation of the $G(\epsilon)$ with conformal mapping

$$G^{(N)}(\epsilon) = \sum_{n=0}^{N} G_n \varepsilon^n \quad \to \quad B^{(N)}(t) = \sum_{n=0}^{N} \frac{G_n}{\Gamma(n+b+1)} t^n \quad \to$$

$$\to \quad \widetilde{B}^{(N)}(t) = \left(\frac{t}{w(t)}\right)^{\lambda} \sum_{n=1}^{N} \widetilde{B}_n w(t)^n$$

λ is another fitting parameter which governs strong coupling asymptotics.

To determine the optimal fitting parameters we optimize the quantity:

$$Q(\lambda, f_2) = \sqrt{\left(\partial_\lambda (f^{(N)}_{\lambda,f_2}) - f^{(N-1)}_{\lambda,f_2})\right)^2 + \left(\partial_\lambda f^{(N)}_{\lambda,f_2}\right)^2}$$

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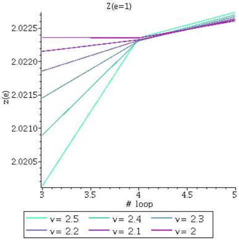
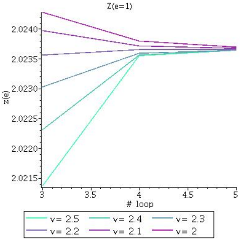
[8] improvement of "boundary condition" method by R.Guida, J.Zinn-Justin: enforce series to coincide with Onsager solution at $\epsilon = 2$

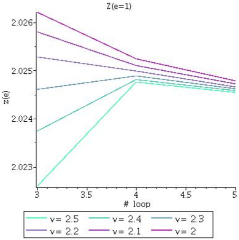
FREE BOUNDARY CONDITION

$$Q(\lambda, f_2) = \sqrt{\left(\partial_\lambda (f^{(N)}_{\lambda, f_2}) - f^{(N-1)}_{\lambda, f_2})\right)^2 + \left(\partial_\lambda f^{(N)}_{\lambda, f_2}\right)^2}$$

Optimization criterion constructed intended to select parameters where we have stable values with accuracy increasing from order to order:

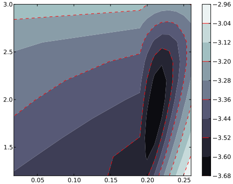




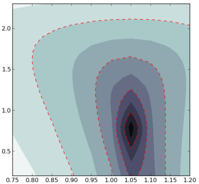


In the fitting parameter space $Q(\lambda, f_2)$ has sharp minimum

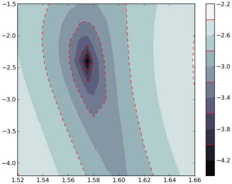
η



$1/\nu$



ω



$\eta^{(6)}_{fbc}(\epsilon = 1) = 0.0355(5)$ $\quad \nu^{(6)}_{fbc}(\epsilon = 1) = 0.629(1)$ $\quad \omega^{(6)}_{fbc}(\epsilon = 1) = 0.8106(4)$

$\eta^{(7)}_{fbc}(\epsilon = 1) = 0.035(1)$ $\quad \nu^{(7)}_{fbc}(\epsilon = 1) = 0.6293(4)$ $\quad \omega^{(7)}_{fbc}(\epsilon = 1) = 0.8125(5)$

$\eta^{(7)}_{fbc}(\epsilon = 2)_{fit} = 0.21(1)$ $\quad \nu^{(6)}_{fbc}(\epsilon = 2)_{fit} = 0.94(4)$ $\quad \omega^{(6)}_{fbc}(\epsilon = 2)_{fit} = 1.560(3)$

$\eta^{(7)}_{fbc}(\epsilon = 2)_{fit} = 0.21(3)$ $\quad \nu^{(7)}_{fbc}(\epsilon = 2)_{fit} = 0.943(9)$ $\quad \omega^{(7)}_{fbc}(\epsilon = 2)_{fit} = 1.575(3)$

FREE BOUNDARY CONDITION. RESULTS

$$\eta^{(6)}_{fbc}(\epsilon=1)=0.0355(5), \qquad \eta^{(6)}_{fbc}(\epsilon=2)_{fit}=0.21(1)$$
$$\eta^{(7)}_{fbc}(\epsilon=1)=0.035(1), \qquad \eta^{(7)}_{fbc}(\epsilon=2)_{fit}=0.21(3)$$
$$\eta^{(6)}_{CM}(\epsilon=1)=0.0362(6), \qquad \eta^{(6)}_{CM}(\epsilon=2)=0.237(27)$$
$$\eta_{CB}(\epsilon=1)=0.03640(60), \qquad \eta_{CB}(\epsilon=2)=0.25$$

$$\nu^{(6)}_{fbc}(\epsilon=1)=0.629(1), \qquad \nu^{(6)}_{fbc}(\epsilon=2)_{fit}=0.94(4)$$
$$\nu^{(7)}_{fbc}(\epsilon=1)=0.6293(4), \qquad \nu^{(7)}_{fbc}(\epsilon=2)_{fit}=0.943(9)$$
$$\nu^{(6)}_{CM}(\epsilon=1)=0.6292(5), \qquad \nu^{(6)}_{CM}(\epsilon=2)=0.952(14)$$
$$\nu_{CB}(\epsilon=1)=0.63005(45), \qquad \nu_{CB}(\epsilon=2)=1$$

$$\omega^{(6)}_{fbc}(\epsilon=1)=0.8106(5), \qquad \omega^{(6)}_{fbc}(\epsilon=2)_{fit}=1.560(3)$$
$$\omega^{(7)}_{fbc}(\epsilon=1)=0.8125(5), \qquad \omega^{(7)}_{fbc}(\epsilon=2)_{fit}=1.575(3)$$
$$\omega^{(6)}_{CM}(\epsilon=1)=0.820(7), \qquad \omega^{(6)}_{CM}(\epsilon=2)=1.71(9)$$
$$\omega_{CB}(\epsilon=1)=0.84(4), \qquad \omega_{CB}(\epsilon=2)=2$$
$$\omega_{theor}(\epsilon=2)=\{4/3, 1.75, 2\}$$

To be fixed: implement more conservative uncertainties

Homographic transformation $\epsilon = \epsilon'/(1+q\epsilon')$

$$f(\epsilon) = \sum_{n=0}^{N} A_n \epsilon^n \quad \to G(\epsilon') = \sum_{n=0}^{N} G_n (\epsilon')^n$$

is intended to soften singularities at large values of ϵ. [9]

$$G^{(N)}(\epsilon') = \sum_{n=0}^{N} G_n (\epsilon')^n \quad \to \quad B^{(N)}(t) = \sum_{n=0}^{N} \frac{G_n}{\Gamma(n+b+1)} t^n \quad \to$$

$$\to \quad \widetilde{B}^{(N)}(t) = \left(\frac{t}{w(t)}\right)^{\lambda} \sum_{n=1}^{N} \widetilde{B}_n w(t)^n$$

We perform optimization over 3 parameters: q, λ and b minimizing error estimate defined as [10]

$$\begin{aligned} E_N^f(b,\lambda,q) \equiv \max\{|\tilde{f}_N^{b,\lambda,q} - \tilde{f}_{N-1}^{b,\lambda,q}|, |\tilde{f}_N^{b,\lambda,q} - \tilde{f}_{N-2}^{b,\lambda,q}|\} \\ + \max\{Var_b\left(\tilde{f}_N^{b,\lambda,q}\right), Var_b\left(\tilde{f}_{N-1}^{b,\lambda,q}\right)\} \\ + Var_\lambda\left(\tilde{f}_N^{b,\lambda,q}\right) + Var_q\left(\tilde{f}_N^{b,\lambda,q}\right). \end{aligned}$$

[9] J.Zinn-Justin

[10] M.Kompaniets, E.Panzer, Phys.Rev.D. 96, 036016 (2017)

$$
\begin{aligned}
E_N^f(b,\lambda,q) \equiv \max\{|\tilde{f}_N^{b,\lambda,q} - \tilde{f}_{N-1}^{b,\lambda,q}|, |\tilde{f}_N^{b,\lambda,q} - \tilde{f}_{N-2}^{b,\lambda,q}|\} \\
+ \max\{Var_b\left(\tilde{f}_N^{b,\lambda,q}\right), Var_b\left(\tilde{f}_{N-1}^{b,\lambda,q}\right)\} \\
+ Var_\lambda\left(\tilde{f}_N^{b,\lambda,q}\right) + Var_q\left(\tilde{f}_N^{b,\lambda,q}\right).
\end{aligned}
$$

Such a complicated error estimation is intended to not underestimate uncertainties of the resummation method and tries to take into account almost all factors.

$$\eta_{CM}^{(6)}(\epsilon=1) = 0.0362(6), \qquad \eta_{CM}^{(6)}(\epsilon=2) = 0.237(27)$$

$$\eta_{CB}(\epsilon=1) = 0.03640(60), \qquad \eta_{CB}(\epsilon=2) = 0.25$$

$$\nu_{CM}^{(6)}(\epsilon=1) = 0.6292(5), \qquad \nu_{CM}^{(6)}(\epsilon=2) = 0.952(14)$$

$$\nu_{CB}(\epsilon=1) = 0.63005(45), \qquad \nu_{CB}(\epsilon=2) = 1$$

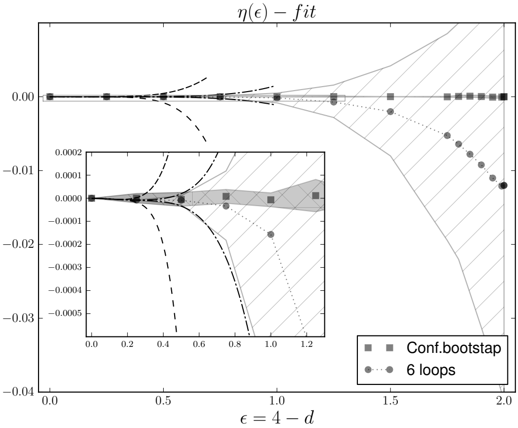
$$\omega_{CM}^{(6)}(\epsilon=1) = 0.820(7), \qquad \omega_{CM}^{(6)}(\epsilon=2) = 1.71(9)$$

$$\omega_{CB}(\epsilon=1) = 0.84(4), \qquad \omega_{CB}(\epsilon=2) = 2$$

$$\omega_{theor}(\epsilon=2) = \{4/3, 1.75, 2\}$$

COMPARISON WITH CONFORMAL BOOTSTRAP

η, $\Delta_\sigma = D/2 - 1 + \eta/2$

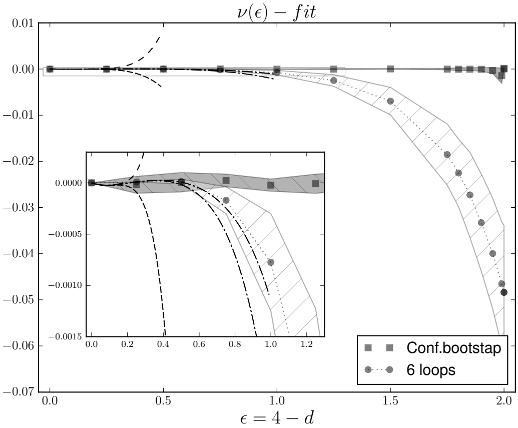


$\eta^{(6)}_{Pade}(\epsilon = 2) = 0.20229(8)$ $\eta^{(7)}_{Pade}(\epsilon = 2) = 0.28(8)$ $\eta^{(6)}_{PB}(\epsilon = 2) = 0.217(0)$

$\eta^{(7)}_{PB}(\epsilon = 2) = 0.244(0)$ $\eta^{(6)}_{Ad}(\epsilon = 2) = 0.2022(8)$ $\eta^{(7)}_{fbc}(\epsilon = 2) = 0.21(3)$

$\eta^{(6)}_{CM}(\epsilon = 2) = 0.237(27)$ $\eta_{CB}(\epsilon = 2) = 0.25$

$\nu,\ \Delta_\epsilon = D - \frac{1}{\nu}$

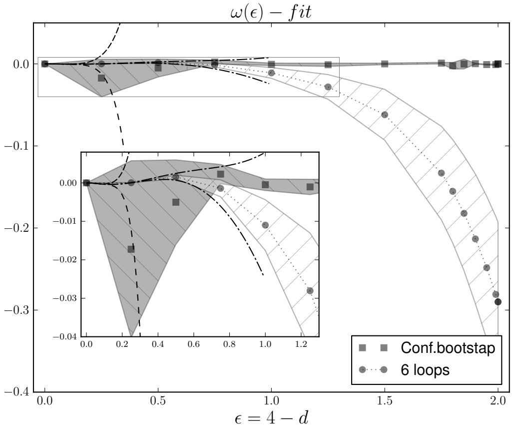


$\nu^{(6)}_{Pade}(\epsilon = 2) = 0.98(4)$ $\quad\nu^{(7)}_{Pade}(\epsilon = 2) = 0.92(3)$ $\quad\nu^{(6)}_{PB}(\epsilon = 2) = 0.904(1)$

$\nu^{(7)}_{PB}(\epsilon = 2) = 1.082(0)$ $\quad\nu^{(6)}_{Ad}(\epsilon = 2) = 0.936(4)$ $\quad\nu^{(7)}_{fbc}(\epsilon = 2) = 0.943(9)$

$\nu^{(6)}_{CM}(\epsilon = 2) = 0.952(14)$ $\quad\nu_{CB}(\epsilon = 2) = 1$

$\omega,\ \Delta_{\epsilon'} = D + \omega$



$\omega^{(6)}_{Pade}(\epsilon = 2) = 1.53164(2)$ $\quad \omega^{(7)}_{Pade}(\epsilon = 2) = 1.9(4)$ $\quad \omega^{(6)}_{PB}(\epsilon = 2) = 1.567(0)$

$\omega^{(7)}_{PB}(\epsilon = 2) = 1.872(0)$ $\quad \omega^{(6)}_{Ad}(\epsilon = 2) = 1.81(12)$ $\quad \omega^{(7)}_{fbc}(\epsilon = 2) = 1.585(3)$

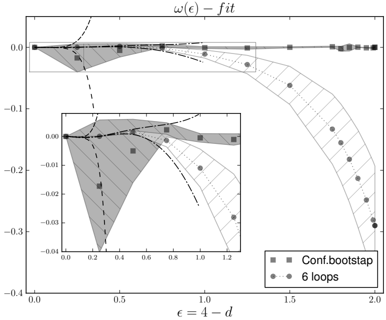
$\omega^{(6)}_{CM}(\epsilon = 2) = 1.71(9)$ $\quad \omega_{CB}(\epsilon = 2) = 2$ $\quad \omega_{theor}(\epsilon = 2) = \{4/3, 1.75, 2\}$

DISCUSSION

DISCUSSION (1/3)

What we got?

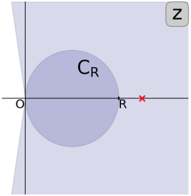
1. We didn't observe any specific behavior of the ϵ-expansion near $d = 2.2$ ($\epsilon = 1.8$) as it was expected from conformal bootstrap study.
2. We observe significant difference in exponents between conformal bootstrap and ϵ-expansion starting from $d = 3.5$ ($\epsilon > 0.5$).
3. Different resummation methods implemented on top of ϵ-expansion provide consistent results which are different from conformal bootstrap.
4. Situation with exponent ω is not completely clear, as for ϕ^4 model contrary to bootstrap prediction $\omega = 2$ there are alternate predictions $\omega = 4/3$ and $\omega = 1.75$



DISCUSSION (2/3)

What may cause such a deviations? And what to do?

1. Usually deviations from Onsager solution are argued to slow convergence due to the large value of the expansion parameter ($\epsilon = 2$).
   As we see deviations starts at $\epsilon \sim 0.75$ which is believed to be small enough.
2. It does not look like problem of the implementation of the resummation algorithm as different methods provide consistent results.
3. Defect of ϵ-expansion?
   3.1 Borel summability does not proven for $d = 4$ (non-summability also)
   3.2 Even if we believe in Borel summability, Socal-Watson theorem guarantied analiticity of the resummed function only inside C_R. But outside C_R it may contain singularities which may to prevent to extend results to the physical values of ϵ.



   3.3 We can implement g-summation:
   ϵ-summ. $\beta(g^*) = 0 \to g^* = \sum g_n \epsilon^n \to \eta = resum_\epsilon(2\gamma_\phi(\sum g_n \epsilon^n))$

   g-summ. $resum_g(\beta)(g^*) = 0 \to g^* = const \to \eta = resum_g(2\gamma_\phi(g))|_{g=g^*}$
   3.4 Renormalization group in fixed space dimensions (noninteger also)

DISCUSSION (3/3)

- 3.5 Convergent expansions:
  Shift expansion point of the continual integral from Gaussian one to make expansion convergent.
  (*Shaverdian, B. and Ushveridze, A., Phys. Lett. 123B, 316-318 (1983)*
  Ivanov, A. and Sazonov, V., Nucl. Phys. B914 43-61 (2017))

4. Conformal bootstrap:
   - 4.1 Investigate origin and properties of the conformal states rearrangement observed at $d = 2.2$. Might be it starts much earlier?
   - 4.2 Investigate in details area $4 > d > 3.25$ ($\epsilon < 0.75$) where ϵ-expansion is expected to work properly.

Thank You!