Critical properties of three-dimensional QED

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Based on works with Sofian Teber and Vadim I. Shilin

AVK, Shilin and Teber, PRD **94** (2016) 056009 [arXiv:1602.01962 [hep-th]]
AVK and Teber, PRD **94** (2016) no.11, 114011 [arXiv:1605.01911 [hep-th]]
AVK and Teber, PRD **94** (2016) 114010 [arXiv:1610.00934 [hep-th]]
AVK and Teber, PRD **99** (2019) 059902 [arXiv:1902.03790 [hep-th]]

Outline

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$(2+1)$-dimensional quantum electrodynamics (QED_3)

Extensive studies for more that three decades now:

- **Original interest:** [Pisarski '84; Appelquist et al. '84] similarities to $(3+1)$-dimensional QCD and toy model to study systematically dynamical chiral symmetry breaking (DχSB)
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"Chiral" (flavour) symmetry in QED$_3$

Massless QED$_3$ with N flavours of 4-component fermions

$$L = \overline{\Psi}_\sigma (i\hat{\partial} - e\hat{A})\Psi^\sigma - \frac{1}{4}F_{\mu\nu}^2 \qquad (\sigma = 1, \cdots, N)$$

Global $U(2N)$ "chiral" (flavour) symmetry

Only possible with 4-component spinors because in this case it is possible to add γ^3 and γ^5 anticommuting with γ^0, γ^1 and γ^2

To see this, let χ_i be a 2-component spinor ($i = 1, \cdots, 2N$). Then:

$$\Psi_\sigma = \begin{pmatrix} \chi_\sigma \\ \chi_{N+\sigma} \end{pmatrix}, \quad \overline{\Psi}_\sigma = \left(\overline{\chi}_\sigma\,, \overline{\chi}_{N+\sigma}\right)\gamma^{35}, \quad \overline{\chi}_\sigma = \chi^\dagger \sigma_3 \quad \gamma^{35} = \gamma^3\gamma^5\,.$$

Fermion bilinears: (parity: $\psi'(-x,y) = \mathrm{i}\gamma^1\gamma^3\psi(x,y)$, $U(2N)$: $\chi'_i = U_i^j \chi_j$)

$\overline{\Psi}_\sigma\,\gamma^\mu\,\Psi^\sigma = \bar{\chi}_i\,\sigma^\mu\,\chi^i$ (parity even, $U(2N)$ invariant)

$\overline{\Psi}_\sigma\,\Psi^\sigma = \bar{\chi}_\sigma\,\chi^\sigma - \bar{\chi}_{N+\sigma}\,\chi^{N+\sigma}$ (parity even, breaks $U(2N)$)

$\overline{\Psi}_\sigma\,\gamma^{35}\,\Psi^\sigma = \bar{\chi}_i\,\chi^i$ (parity odd, $U(2N)$ invariant)

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Parity-even mass term breaks $U(2N) \to U(N) \times U(N)$

$$\overline{\Psi}_\sigma \, \Psi^\sigma = \bar{\chi}_\sigma \, \chi^\sigma - \bar{\chi}_{N+\sigma} \, \chi^{N+\sigma}$$

Question (parity-odd mass term neglected): [Pisarski ’84]

Is it possible that “chiral” symmetry is dynamically broken in QED$_3$?

(with dynamical generation of a parity-even mass)

Some properties of the model

QED_3 is super-renormalizable

- dimensionful coupling constant $a = Ne^2/8$
- loop-expansion plagued by IR singularities (starting from two-loop) [Jackiw & Templeton '81] [Guendelman & Radulovic '83, '84]

Large$-N$ limit of QED_3 ($N \to \infty$ and a fixed): **IR softening**

[Appelquist & Pisarski '81, Appelquist & Heinz '81]

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2\,[1+\Pi(p)]} = \frac{g_{\mu\nu}}{p^2\,[1+a/|p|]} \xrightarrow{p \ll a} \frac{g_{\mu\nu}}{a|p|}$$

- Gauge propagator $\sim 1/p$: Coulomb-like (no confinement)
  Similar to reduced QED [Teber '12, AVK & Teber '13]
- Effective dimensionless coupling:

$$\bar{a}(p) = \frac{a}{|p|(1+a/|p|)} = \begin{cases} 0 & p \gg a \quad \text{free stable UV fixed point} \\ 1 & p \ll a \quad \text{non-trivial IR fixed point} \end{cases}$$

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Power counting in large-N QED_3 similar to 4-dimensional theories

Gauge propagator $\sim 1/p$ and dimensionless coupling $\sim 1/\sqrt{N}$

- model becomes IR finite
- but also UV finite (no renormalization of the gauge field)

scale (conformal) invariance!

Dynamical chiral symmetry breaking in QED_3

- should take place at momentum scales $p \ll a$ (breaks scale invariance)
- cannot take place at any finite order in $1/N$ (requires a non-perturbative approach)
- may take place below some critical fermion flavour number N_c

N_c

$N = 0$ ——— massive ——— N_c ——— massless ——— $N \to \infty$

Challenge: determine the value of N_c

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Value of N_c crucial to understand the phase structure of QED_3
(important consequences for particle/condensed matter physics)

- [Pisarski '84] DχSB for all values of N ($N_c \to \infty$)
  - Method: solves Schwinger-Dyson (SD) gap equation at leading order (LO) of $1/N$-expansion
  - Support: RG study [Pisarski '91], further SD studies [Pennington et al. '91, '92], lattice simulation [Azcoiti '93, '96]

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In 30 years: very different results obtained for N_c!

N_c	**Method**	**Year**
∞	SD (LO in $1/N$)	1984
∞	SD (non-perturbative)	1990, 1992
∞	RG study	1991
∞	lattice simulations	1993, 1996
< 4.4	F-theorem	2015
3.5 ± 0.5	lattice simulations	1988, 1989
$32/\pi^2 \approx 3.24$	SD (LO, Landau gauge)	1988
2.89	RG study (one-loop)	2016
$1 + \sqrt{2} = 2.41$	F-theorem	2016
$< 9/4 = 2.25$	RG study (one-loop)	2015
$< 3/2$	Free energy constraint	1999
0	SD (non-perturbative)	1990
0	lattice simulations	2015, 2016

Table: Values of N_c obtained over the years with different methods (at LO).

Beyond leading order

All these very different results reflect a poor understanding of the problem

[Pisarski '91]: "difficult to presume we understand χSB in QCD$_4$ when we do not fully understand flavour-symmetry breaking in QED$_3$"

Important question: stability of the critical point

Consider the approach of [Appelquist et al. '88] (LO in $1/N$-expansion):

$$N_c = 32/\pi^2 = 3.24 \text{ is not large}$$

Contribution of higher order corrections may be essential!

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SD gap equation at NLO in the $1/N$-expansion

First and most well-known study: [Nash '89]

- non-local ξ-gauge
- additional resummation (of wave function renormalization)

$$\text{LO + resummation: } N_c = (4/3)(32/\pi^2) = 4.32$$

  - fully gauge-invariant
  - 33% deviation wrt to LO in Landau gauge without resummation

- attempt to compute NLO corrections
  - approximate calculation of diagrams
  - different gauges used for different parts of the calculation
  - typo in the final result as noticed by [Gusynin & Pyatkovskiy '16]

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Fermion propagator ($\Sigma(p)$: dynamically generated (parity-conserving) mass, $A(p)$: fermion wave function):

$$S^{-1}(p) = [1 + A(p)]\,(\mathrm{i}\hat{p} + \Sigma(p))$$

Photon propagator (non-local ξ-gauge, $\xi = 0$: Landau gauge):

$$D_{\mu\nu}(p) = \frac{P^{\xi}_{\mu\nu}(p)}{p^2\,[1 + \Pi(p)]}, \quad P^{\xi}_{\mu\nu}(p) = g_{\mu\nu} - (1 - \xi)\frac{p_\mu p_\nu}{p^2}\,,$$

SD equation for the fermion propagator ($\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$):

$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathrm{Tr}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\gamma^\mu\, D_{\mu\nu}(p-k)\,\Sigma(k)\,\Gamma^\nu(p,k)}{[1 + A(k)]\,(k^2 + \Sigma^2(k))}$$

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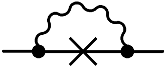
$\Gamma^\nu(p,k)$: vertex function.

At leading order ($a = Ne^2/8$): $A(p) = 0, \quad \Pi(p) = \frac{a}{|p|}, \quad \Gamma^\nu(p,k) = \gamma^\nu$

A single diagram contributes to the gap equation (cross = mass insertion):

$\Sigma(p) =$  $= \dfrac{8(2+\xi)a}{N} \displaystyle\int \frac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[(p-k)^2 + a\,|p-k|\right]}$

Following [Appelquist et al. '88] and [AVK '93]:

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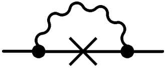
$\Sigma(p) =$ 

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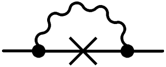
$\Sigma(p) =$  $$= \frac{8(2+\xi)a}{N} \int \frac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{\left(k^2 + \Sigma^2(k)\right) \left[(p-k)^2 + a\,|p-k|\right]}$$

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$$1 = \frac{(2+\xi)\beta}{L} \quad \text{where} \quad \beta = \frac{1}{\alpha\,(1/2-\alpha)} \quad \text{and} \quad L \equiv \pi^2 N$$

Solving the gap equation, yields (in agreement with [Appelquist et al. '88]):

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$$A(p) = \frac{\Gamma(1+\varepsilon)(4\pi)^\varepsilon\mu^{2\varepsilon}}{p^{2\varepsilon}}\,C_1(\xi) = \frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,C_1(\xi) + \mathrm{O}(\varepsilon)$$

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Outline

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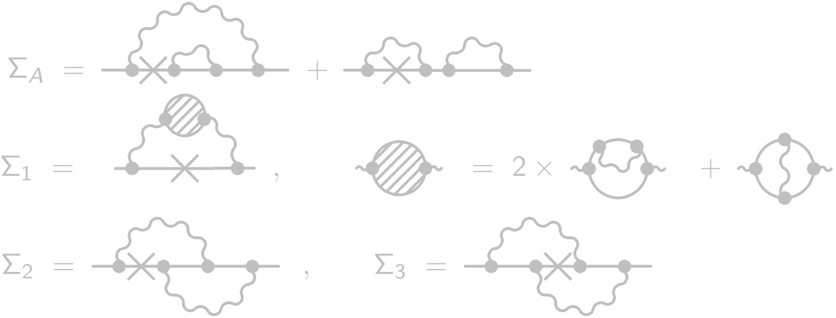

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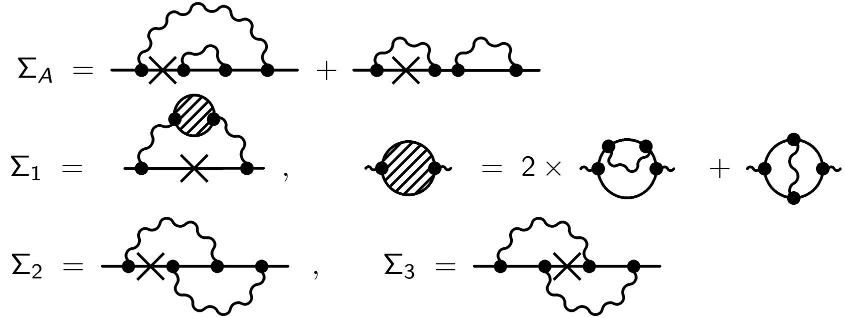

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very powerful and effecient calculational framework

For the year 2016-2017, breakthrough achievements ($+$ beautiful mathematics):

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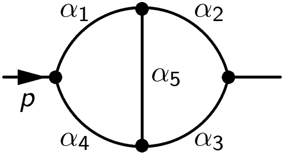
Massless propagator type 2-loop diagram

Basic building block of multi-loop calculations ($[\mathrm{d}^D k] = \mathrm{d}^D k/(2\pi)^D$):

$$J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \int\int \frac{[\mathrm{d}^D k_1]\,[\mathrm{d}^D k_2]}{k_1^{2\alpha_1}\, k_2^{2\alpha_2}\,(k_2-p)^{2\alpha_3}\,(k_1-p)^{2\alpha_4}\,(k_2-k_1)^{2\alpha_5}}$$

Arbitrary indices α_i and external momentum p in Euclidean space (D)



Coefficient function (dimensionless as our Σ_i):

$$G(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \frac{(4\pi)^D}{(p^2)^{D-\sum_{i=1}^5 \alpha_i}}\, J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$$

Goal of multi-loop computation:

in $D = n - 2\varepsilon$ ($n=3$), compute $G(\{\alpha_i\})$ as a Laurent series in $\varepsilon \to 0$

Long history: exact computations are crucial

- all indices integers: well-known and easy to compute, *e.g.* IBP [Vasil'ev, Pismak & Khonkonen '81] [Tkachov '81] [Chetyrkin & Tkachov '81]
- $\alpha_i = n_i + a_i\varepsilon$ $(\forall i)$: already non-trivial [Kazakov '83, '84, '85] [Broadhurst '86, '03] Automated in [Bierenbaum & Weinzierl '03] ($D = 4$: MZV)

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Idea of the method (algebraic, no explicit integration):

compute complicated Feynman diagrams
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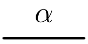

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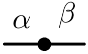

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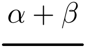
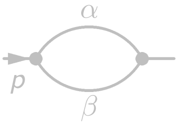


- Simple loops involve an integration (used in previous LO calculation):

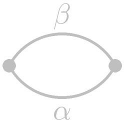
$$\int \frac{[\mathrm{d}^D k]}{k^{2\alpha}(p-k)^{2\beta}} =$$



$$= \frac{(p^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}}\, G(\alpha,\beta)$$

where

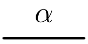
$$G(\alpha,\beta) =$$



$$= \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$$

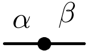
Some rules (in momentum space):

- Plain line with an arbitrary index α:

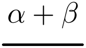


$$\Longleftrightarrow \quad \frac{1}{k^{2\alpha}}$$

- Chains reduce to the product of propagators:



$$=$$



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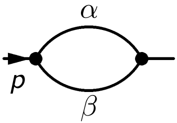
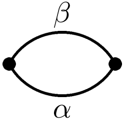


$$= \frac{(p^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}}\, G(\alpha,\beta)$$

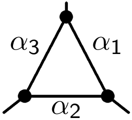
where

$$G(\alpha,\beta) =$$

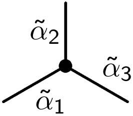


$$= \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$$

- Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$):



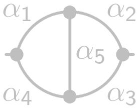
$$\underset{\sum_i \alpha_i = D}{=} \frac{G(\alpha_1, \alpha_2)}{(4\pi)^{D/2}}$$



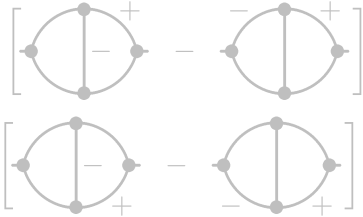
(Note: unique triangle has index $\sum_i \alpha_i = D$)

- IBP relation:

$(D - \alpha_2 - \alpha_3 - 2\alpha_5)$



$= \alpha_2$ ... $+ \alpha_3$



(Note: $\pm$ corresponds to add or subtract 1 to index α_i)

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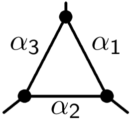

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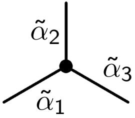

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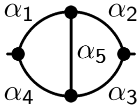
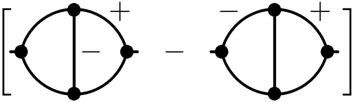
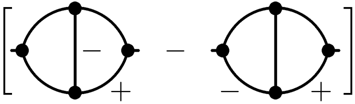


$= \ \alpha_2$

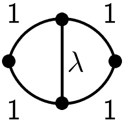


$+ \ \alpha_3$



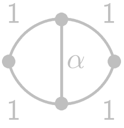
(Note: $\pm$ corresponds to add or subtract 1 to index α_i)

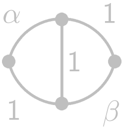
Example: this allows to compute [Vasil'ev et al. '81] [Vasil'ev et al. '93] [Kivel et al. '94] [AVK & Teber '13] ($\lambda = D/2 - 1$, $\lambda = 1/2$ in $D = 3$)

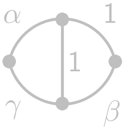
$G(1,1,1,1,\lambda) =$ 

$= 3\,\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}\left[\Psi'(\lambda) - \Psi'(1)\right]$

But the following diagrams are beyond IBP and uniqueness:







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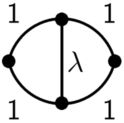
$G(1,1,1,1,\lambda) =$ 

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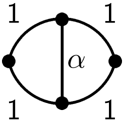
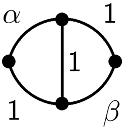
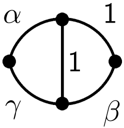
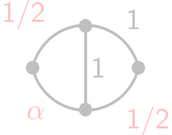
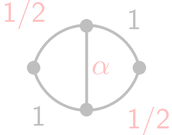






... and our calculations require the evaluations of masters such as:

$\tilde{I}_1(\alpha) =$ 

$\tilde{I}_2(\alpha) =$ 

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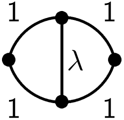

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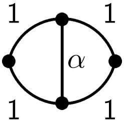

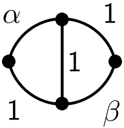

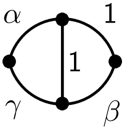

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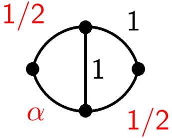

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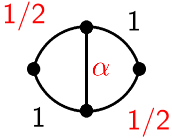

... for which no exact solution is still available.

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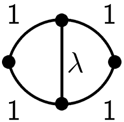

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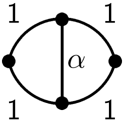

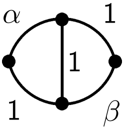

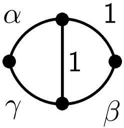

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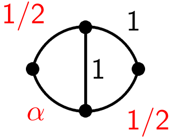

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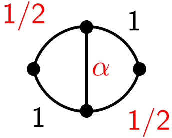

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Technicalities (2)

The Gegenbauer polynomial x-space technique
[Chetyrkin, Kataev & Tkachov '80] [AVK '96]

Gegenbauer polynomial C_n^β of degree n and index β defined as:

$$\frac{1}{(1-2xw+w^2)^\beta} = \sum_{k=0}^{\infty} C_k^\beta(x)\, w^k \qquad C_n^\beta(1) = \frac{\Gamma(n+2\beta)}{\Gamma(2\beta)\, n!}$$

Orthogonality relation on the unit D-dimensional sphere ($\hat{x} = x/\sqrt{x^2}$):

$$\frac{1}{\Omega_D} \int \mathrm{d}_D\, \hat{x}\, C_n^\lambda(\hat{z}\cdot\hat{x}) C_m^\lambda(\hat{x}\cdot\hat{z}) = \delta_{n,m}\, \frac{\lambda\, \Gamma(n+2\lambda)}{\Gamma(2\lambda)\,(n+\lambda)\, n!}, \qquad \lambda = \frac{D}{2} - 1\,,$$

$$\mathrm{d}^D x = \frac{1}{2}\, x^{2\lambda}\, \mathrm{d}x^2\, \mathrm{d}_D \hat{x} \qquad \Omega_D = 2\pi^{D/2}/\Gamma(D/2)$$

They allow to generalize the multi-pole expansion to arbitrary dimension D

For a propagator with arbitrary power β ($\Theta(x) \equiv$ Heaviside (step) function):

$$\frac{1}{(x_1 - x_2)^{2\beta}} = \sum_{n=0}^{\infty} C_n^{\beta}(\hat{x}_1 \cdot \hat{x}_2) \left[\frac{(x_1^2)^{n/2}}{(x_2^2)^{n/2+\beta}} \Theta(x_2^2 - x_1^2) + (x_1^2 \longleftrightarrow x_2^2) \right],$$

where:

$$C_n^{\delta}(x) = \sum_{k=0}^{[n/2]} C_{n-2k}^{\lambda}(x) \frac{(n-2k+\lambda)\Gamma(\lambda)}{k!\,\Gamma(\delta)} \frac{\Gamma(n+\delta-k)\Gamma(k+\delta-\lambda)}{\Gamma(n-k+\lambda+1)\Gamma(\delta-\lambda)}$$

Rules for integrating diagrams with Heaviside functions [AVK '96]

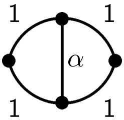
$$\int \frac{\mathrm{d}^D x}{x^{2\alpha}(x-y)^{2\beta}} \Theta(x^2 - y^2) = \frac{\pi^{D/2}}{(y^2)^{\alpha+\beta-\lambda-1}} \sum_{m=0}^{\infty} \frac{B(m,n|\beta,\lambda)}{m+\alpha+\beta-1-\lambda}$$

$$\overset{(\beta=\lambda)}{=} \frac{\pi^{D/2}}{(y^2)^{\alpha-1}} \frac{1}{\Gamma(\lambda)} \frac{1}{(\alpha-1)(n+\lambda)} \qquad \text{(as one example)}$$

$$B(m,n|\beta,\lambda) = \frac{\Gamma(m+n+\beta)}{m!\Gamma(m+n+\lambda+1)\Gamma(\beta)} \frac{\Gamma(m+\beta-\lambda)}{\Gamma(\beta-\lambda)}.$$

With these rules, one-fold series (${}_3F_2$-hypergeometric function of argument 1) obtained [AVK '96]:

$$G(1,1,1,1,\alpha) \quad = $$



$$= \; -2\,\Gamma(\lambda)\Gamma(\lambda-\alpha)\Gamma(1-2\lambda+\alpha)\times$$

$$\times\left[\frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)\Gamma(n+1)}{n!\,\Gamma(n+1+\alpha)}\,\frac{1}{n+1-\lambda+\alpha}+\frac{\pi\cot\pi(2\lambda-\alpha)}{\Gamma(2\lambda)}\right]$$

One-fold series (two ${}_3F_2$-hypergeometric functions of argument -1) obtained earlier by [Kazakov '84] (using functional relations):

$$G(1,1,1,1,\alpha) = -2\,\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon-\alpha)\Gamma(\alpha+2\varepsilon)}{\Gamma(2-2\varepsilon)}\left[\frac{1}{\Gamma(1+\alpha)\Gamma(1-3\varepsilon-\alpha)}\right.$$

$$\left.\times\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}\left(\frac{1}{n+\alpha+\varepsilon}+\frac{1}{n-\alpha-2\varepsilon}\right)+\cos[\pi\varepsilon]\right]$$

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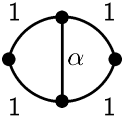

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Recently, the two results were proved to be equal [AVK & Teber '16]

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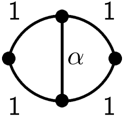

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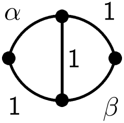
Recently, the two results were proved to be equal [AVK & Teber '16]

In a more complicated case, one-fold series (as a combination of two ${}_3F_2$-hypergeometric functions of argument 1) were also obtained in [AVK & Teber '14] from the rules of [AVK '96]:

$$I(\alpha, 1, \beta, 1, 1) \quad =$$



$$= \quad \frac{1}{\pi^D} \frac{1}{\tilde{\alpha} - 1} \frac{1}{1 - \tilde{\beta}} \times$$

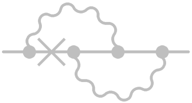
$$\times \quad \frac{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})}{\Gamma(\alpha)\Gamma(\lambda - 2 + \tilde{\alpha} + \tilde{\beta})} \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} \, I(\tilde{\alpha}, \tilde{\beta})$$

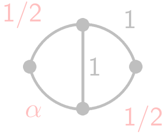
where:

$$I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi\tilde{\alpha}]}{\sin[\pi(\lambda - 1 + \tilde{\beta})] \sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]}$$
$$+ \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!} \Bigg(\frac{1}{n + \lambda + \tilde{\alpha} - 1} \frac{\Gamma(n+1)}{\Gamma(n + 2 + \lambda - \tilde{\beta})} + \frac{1}{n + \lambda + 1 - \tilde{\alpha}} \times$$
$$\times \frac{\Gamma(n + 2 - \tilde{\alpha})\Gamma(2 - \tilde{\beta})\Gamma(\lambda)}{\Gamma(n + 3 + \lambda - \tilde{\alpha} - \tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)} \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]} \Bigg)$$

Back to QED$_3$: for the most complicated integrals, the rules [AVK '96] yield multiple-series representations [AVK, Shilin & Teber '16]

For Σ_2, related master integral represented in terms of a two-fold series

$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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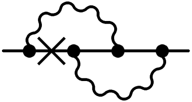
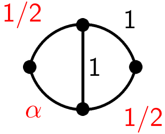
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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$$\tilde{I}_1(\alpha) = \frac{(4\pi)^3}{(p^2)^{-\alpha}} \int \frac{[\mathrm{d}^3 k_1][\mathrm{d}^3 k_2]}{|p-k_1| k_1^{2\alpha} (k_1-k_2)^2 (p-k_2)^2 |k_2|} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{B(l,n,1,1/2)}{(n+1/2)\,\Gamma(1/2)}$$

$$\times \left[\frac{2}{n+1/2} \left(\frac{1}{l+n+\alpha} + \frac{1}{l+n+3/2-\alpha} \right) + \frac{1}{(l+n+\alpha)^2} + \frac{1}{(l+n+3/2-\alpha)^2} \right]$$

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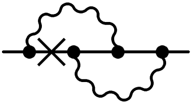
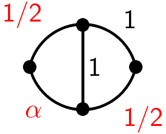
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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Moreover, it obeys the following functional relation:

$$\tilde{I}_1(\alpha+1) = \frac{(\alpha-1/2)^2}{\alpha^2} \tilde{I}_1(\alpha) - \frac{1}{\pi\alpha^2} \left[\Psi'(\alpha) - \Psi'(1/2-\alpha) \right].$$

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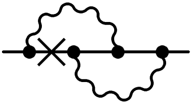
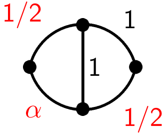
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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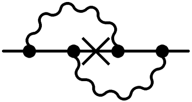
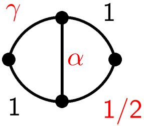
$\Sigma_3 =$ 

$\tilde{I}(\alpha,\gamma) =$ 

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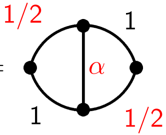
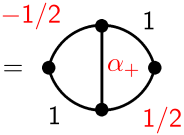
$\tilde{I}_2(\alpha) =$ 

$\tilde{I}_3(\alpha) =$ 

$(\alpha_+ = 1 + \alpha)$

Only one is independent. Functional relations:

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2-\alpha), \quad \tilde{I}_3(\alpha) = \frac{2}{4\alpha-1}\Big(\alpha \tilde{I}_2(1+\alpha) - (1/2-\alpha)\tilde{I}_2(\alpha)\Big) - \frac{\beta^2}{\pi}$$

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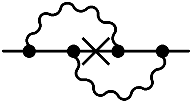
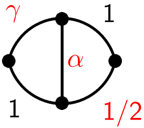
$\Sigma_3 =$ 

$\tilde{I}(\alpha,\gamma) =$ 

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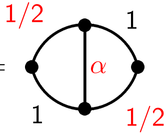
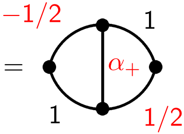
$\tilde{I}_2(\alpha) =$ 

$\tilde{I}_3(\alpha) =$ 

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Representation of $\tilde{I}_2(\alpha)$ in terms of a three-fold series

$$\tilde{I}_2(\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B(m,n,\beta,1/2)\sum_{l=0}^{\infty} B(l,n,1,1/2) \times C(n,m,l,\alpha)\,,$$

$$C(n,m,l,\alpha) = \frac{1}{(m+n+\alpha)(l+n+\alpha)} + \frac{1}{(m+n+\alpha)(l+m+n+1)}$$
$$+\frac{1}{(m+n+1/2)(l+m+n+\alpha)} + \frac{1}{(m+n+1/2)(l+n+3/2-\alpha)}$$
$$+\frac{1}{(n+l+\alpha)(l+m+n+\alpha)} + \frac{1}{(l+n+3/2-\alpha)(l+n+m+\alpha)}\,.$$

Back to the gap equation at NLO

$$1 = \frac{(2+\xi)\beta}{L} + \frac{\overline{\Sigma}_A(\xi) + \overline{\Sigma}_1(\xi) + 2\,\overline{\Sigma}_2(\xi) + \overline{\Sigma}_3(\xi)}{L^2}$$

All diagrams can be computed exactly

Contribution of $\overline{\Sigma}_A$ originates from LO $A(p)$ (singular):

$$\overline{\Sigma}_A(\xi) = 4\,\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left[\left(\frac{4}{3}(1-\xi)-\xi^2\right)\left[\frac{1}{\varepsilon}+\Psi_1-\frac{\beta}{4}\right]+\left(\frac{16}{9}-\frac{4}{9}\xi-2\xi^2\right)\right]$$

where $\Psi_1 = \Psi(\alpha) + \Psi(1/2-\alpha) - 2\Psi(1) + \dfrac{3}{1/2-\alpha} - 2\ln 2$

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$$\overline{\Sigma}_1(\xi) = -2(2+\xi)\,\beta\,\hat{\Pi}, \qquad \hat{\Pi} = \frac{92}{9}-\pi^2\,,$$

Notice: ξ-dependence comes from the fact that we work in a non-local gauge

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The contribution $\overline{\Sigma}_2$ is singular:

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$$\left.+\frac{28}{9}+\frac{8}{9}\,\xi-4\xi^2\right]+(1-\xi)\,\hat{\Sigma}_2\,,$$

where $\hat{\Sigma}_2$ is the "complicated" part (depending on $\tilde{I}_1(\alpha)$):

$$\hat{\Sigma}_2(\alpha) = (4\alpha-1)\beta\Big[\Psi'(\alpha)-\Psi'(1/2-\alpha)\Big]+\frac{\pi\,\tilde{I}_1(\alpha)}{2\alpha}+\frac{\pi\,\tilde{I}_1(\alpha+1)}{2(1/2-\alpha)}\,.$$

Singularities in $\overline{\Sigma}_A(\xi)$ and $\overline{\Sigma}_2(\xi)$ cancel each other and the sum is finite:

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Finally, the contribution of $\overline{\Sigma}_3$ is finite too:

$$\overline{\Sigma}_3(\xi) = \hat{\Sigma}_3(\alpha,\xi) + \Big(3 + 4\xi - 2\xi^2\Big)\beta^2\,,$$

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$$\begin{aligned}\hat{\Sigma}_3(\alpha,\xi) = &\frac{1}{4}\big(1 + 8\xi + \xi^2 + 2\alpha(1-\xi^2)\big)\pi\tilde{I}_2(\alpha)\\ &+ \frac{1}{2}\big(1 + 4\xi - \alpha(1-\xi^2)\big)\pi\tilde{I}_2(1+\alpha)\\ &+ \frac{1}{4}\big(-7 - 16\xi + 3\xi^2\big)\pi\tilde{I}_3(\alpha)\,.\end{aligned}$$

Notice: while $\hat{\Sigma}_2$ is gauge-invariant, $\hat{\Sigma}_3$ depends on ξ

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Gap equation (1)

Combing all previous results yields the exact gap equation:

$$\boxed{\begin{aligned}1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[&8S(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta - \left(\frac{5}{3} - \frac{26}{3}\xi + 3\xi^2\right)\beta^2 \\ &- 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\Big],\end{aligned}}$$

where $S(\alpha,\xi)$ contains all the "complicated" parts:

$$\boxed{S(\alpha,\xi) = \left(\hat{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\hat{\Sigma}_2(\alpha)\right)/8\,.}$$

Gap equation (2)

Previous results show that LO $\sim \beta$ while NLO has $\sim \beta$ and $\sim \beta^2$ contribution:

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$$\left. - 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\right]$$

Extracting terms $\sim \beta$ and $\sim \beta^2$ from the "complicated" part:

$$S(\alpha,\xi) = \left(\hat{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\hat{\Sigma}_2(\alpha)\right)/8\,.$$

yields another, equivalent, gap equation:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\,\beta^2 \right.$$
$$\left. +4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right],$$

where $\quad \tilde{S}(\alpha,\xi) = \left(\tilde{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\tilde{\Sigma}_2(\alpha)\right)/8 \quad$ is "the rest" .

Nash's resummation

The last form of the gap equation

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\,\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big],$$

is a convenient starting point to implement a resummation of the wave-function renormalization constant [Nash '89]

Recall that ($\lambda^{(1)}$ at LO and $\lambda^{(2)}$ at NLO from [Gracey '93]):

$$\lambda_A = \frac{\lambda^{(1)}}{L} + \frac{\lambda^{(2)}}{L^2} + \cdots, \quad \lambda^{(1)} = 4\left(\frac{2}{3} - \xi\right), \quad \lambda^{(2)} = -8\left(\frac{8}{27} + \left(\frac{2}{3} - \xi\right)\hat{\Pi}\right)$$

Crucial observation: the NLO term $\sim \beta^2$ is proportional to $\lambda^{(1)}$

(in the gap equation the LO and NLO contain, respectively, the zeroth and first-order terms in λ_A)

$\Rightarrow$ resum the full expansion of λ_A at the level of the gap equation!

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For more details, the beautiful observation of [Nash '89] is that the gap equation

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\,\beta^2 \\ +4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big]\,,$$

can be re-written (in Appelquist form [Appelquist et al. '88]):

$$1 = \frac{4(2+\xi)}{L\Sigma(p)}\int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left\{1 + \frac{4(2-3\xi)}{3L}\ln\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]\right\} + \frac{\Delta(\alpha,\xi)}{L^2}\,,$$

where $\quad \Delta(\alpha,\xi) = 8\tilde{S}(\alpha,\xi) - 4\beta\left(\xi^2 + 4\xi + \frac{8}{3} + \frac{2+\xi}{2}\hat{\Pi}\right)$.

For resummation:

$$\int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left\{1 + \frac{\lambda^{(1)}}{L}\ln\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]\right\} \to \int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]^{\lambda_A}$$

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where the LO term is now gauge independent [Nash '89].

More explicitly, our careful analysis shows that:

$$1 = \frac{8\beta}{3L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\,\beta\,\left(\frac{40}{9}+\hat{\Pi}\right)\right],$$

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Solve for α (2 standard asymptotics of $\Sigma(k) = B\,(k^2)^{-\alpha_\pm}$):

$$\alpha_\pm = \frac{1}{4}\,\left(1 \pm \sqrt{1 - \frac{128}{3L} + \frac{256}{3L^2}\,\left(\frac{40}{9} + \hat{\Pi}\right)}\right)$$

Expansion of $\alpha_\pm$ in $1/L$ yields:

$$\alpha_- = \frac{16}{3L} + \frac{32}{3L^2}\,\left(\pi^2 - \frac{28}{3}\right) + \mathrm{O}(1/L^3), \qquad \alpha_+ = \frac{1}{2} - \alpha_-\,.$$

It turns out that: $2\alpha_- = \gamma_m(L)$ which corresponds to the $1/L^2$ mass anomalous dimension of [Gracey '93]

For dynamical (wrt explicit) χSB, only one UV asymptotics appears:

$$\Sigma(k) \sim p^{1-\gamma_m(L)} = p^{-2\alpha_+(L)}$$

such that $\alpha_+(L)$ becomes complex for $\overline{L}_c = 28.0981$ or $\overline{N}_c = 2.85$

Outline

Reduced $\text{QED}_{4,3}$

Fermion field in $2+1$-dimensions and photon field in $3+1$-dimensions:

$$S = \int \mathrm{d}^3x \, \bar{\psi}_\sigma \mathrm{i} \not{D} \psi^\sigma + \int \mathrm{d}^4x \, \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]$$

Boundary effective Lagrangian (in 3 dimensions): non-local

$$L = \bar{\psi}_\sigma \mathrm{i} \left(\not{\partial} + \mathrm{i} e \tilde{\not{A}} \right) \psi^\sigma - \frac{1}{4} \tilde{F}^{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\tilde{\xi}} \tilde{A}^\mu \frac{2\,\partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}^\nu$$

$\tilde{\xi} = (1+\xi)/2$: gf parameter associated to reduced gauge field $\tilde{A}^\mu$

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General graphene model (massless, with retardation $x = v/c$):

$$S = \int \mathrm{d}t\, \mathrm{d}^2 x \left[\bar{\psi}_\sigma \left(\mathrm{i}\gamma^0 \partial_t + \mathrm{i} v \vec{\gamma} \cdot \vec{\nabla} \right) \psi^\sigma - e \bar{\psi}_\sigma \, \gamma^0 A_0 \, \psi^\sigma + e \frac{v}{c} \, \bar{\psi}_\sigma \, \vec{\gamma} \cdot \vec{A} \, \psi^\sigma \right]$$
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IR fixed point: running v [Gonzalez, Guinea & Vozmediano '94]

$$\beta_v(\alpha_g) = \begin{cases} -\frac{v\,\alpha_g}{4} \left(1 - \frac{1}{2}\, x^2 + \mathrm{O}(x^3)\right) & \text{(case } x \to 0) \\ -\frac{8(1-x)\, v\, \alpha_g}{5\pi} \left(1 - \frac{19}{42}\,(1-x) + \mathrm{O}((1-x)^2)\right) & \text{(case } x \to 1) \end{cases}$$

such that

$$\begin{cases} v(\mu = 200\text{meV}) \approx c/300 & \underset{\mu \to 0}{\longrightarrow} \quad c \\ \alpha_g(\mu = 200\text{meV}) = e^2/(4\pi\hbar v) \approx 2.2 & \underset{\mu \to 0}{\longrightarrow} \quad \alpha_{QED} = 1/137 \end{cases}$$

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- $U(4)$ invariance ($N_g = 2$ for graphene): sublattice symmetry
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Last 10 years: different results obtained for α_c!

α_c (N_c)	**Method**	**Year**
7.65	SD (LO, dynamic RPA, running v)	2013
3.7	FRG, Bethe-Salpeter	2016
$3.2 < \alpha_c < 3.3$	SD (LO, dynamic RPA, running v)	2012
3.1	SD (LO, bare vertex approximation)	2015
2.06	SD (LO, dynamic RPA, running v)	2017
1.62	SD (LO, static RPA)	2002
(3.52)	SD (LO)	2009
1.13 (3.6)	SD (LO, static RPA, running v)	2008
1.11 ± 0.06	Lattice simulations	2008
0.99	RG study	2012
0.92	SD (LO, dynamic RPA)	2009
0.9 ± 0.2	Lattice simulations	2012
0.833	RG study	2008

Table: Values of α_c and N_c obtained over the years with different methods.

Our approach: focus on the deep IR Lorentz-invariant fixed point where the system is described by $\mathrm{QED}_{4,3}$ (ultra-relativistic limit with fully retarded interactions $x = v/c \to 1$ and $\alpha = 1/137$)

Techniques of massless Feynman diagram calculations apply: attempt to reach a quantitative understanding of the effect of electron-electron interactions in this (academic) limit [Teber '12]

Many recent works on $\mathrm{QED}_{4,3}$: quantum Hall physics [Marino et al. '14, '15], optical properties [Raya et al. '15, '16], 1/2-filled FQHE systems [Son '15], LKF [Ahmad et al. '16], duality [Hsiao & Son '17], bCFT [Herzog & Huang '17]...

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- laboratory to study DχSB in lower dimensional brane theories
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Mapping [AVK & Teber '16]

large-N QED$_3$ (coupling $\sim 1/N$) and QED$_{4,3}$ (coupling $\alpha = e^2/(4\pi)$)

Origin of the mapping: photon propagators have the same form

$$D^{\mu\nu}_{\mathrm{RQED}}(p) = \frac{d^{\mu\nu}(\eta/2)}{2\,|p|}, \quad D^{\mu\nu}_{\mathrm{QED3}}(p) = \frac{d^{\mu\nu}(\tilde\eta)}{a|p|} \quad d^{\mu\nu}(\eta) = g^{\mu\nu} - \eta\frac{p^\mu p^\nu}{p^2}$$

Both theories have power counting similar to four-dimensional ones and are scale (conformal) invariant

Transformations (from large-N QED to reduced QED$_{4,3}$)

$$\frac{1}{L} \equiv \frac{1}{\pi^2 N} \to \frac{\alpha}{4\pi} \equiv \frac{e^2}{(4\pi)^2}, \qquad \tilde\eta \to \frac{\eta}{2} \quad \left(\tilde\xi \to \frac{1+\xi}{2}\right)$$

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A check: from Gracey's result [Gracey '93] for the NLO fermion anomalous dimension in QED_3 we recover the result of [AVK & Teber '14]

$$\lambda_\psi = \frac{4}{L}\left(\frac{2}{3} - \tilde{\xi}\right) - \frac{8}{L^2}\left(\frac{8}{27} + \left(\frac{2}{3} - \tilde{\xi}\right)\hat{\Pi}_2\right) + \mathrm{O}(1/L^3)$$
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For DχSB: while N is discrete (integer), α is continuous (real)
At NLO + Nash's resummation + RPA (using Gusynin's prescription):

$$\overline{\alpha}_c(N=2) = 1.22, \qquad \overline{\alpha}_c(N=1) = 0.6229$$
$$\overline{N}_c = 3.04$$

Surprisingly α_c is of the same order as in the non-relativistic limit...

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For DχSB: while N is discrete (integer), α is continuous (real)
At NLO + Nash's resummation + RPA (using Gusynin's prescription):

$$\overline{\alpha}_c(N=2) = 1.22, \qquad \overline{\alpha}_c(N=1) = 0.6229$$
$$\overline{N}_c = 3.04$$

Surprisingly α_c is of the same order as in the non-relativistic limit...

α_c (N_c)	**Method**	**Year**
7.65	SD (LO, dynamic RPA, running v)	2013
3.7	FRG, Bethe-Salpeter	2016
$3.2 < \alpha_c < 3.3$	SD (LO, dynamic RPA, running v)	2012
3.1	SD (LO, bare vertex approximation)	2015
2.06	SD (LO, dynamic RPA, running v)	2017
1.62	SD (LO, static RPA)	2002
$\alpha_c = 1.22$ $N_c = 3.04$	SD (NLO, RPA, resummation, $v/c \to 1$)	2016
(3.52)	SD (LO)	2009
1.13 (3.6)	SD (LO, static RPA, running v)	2008
1.11 ± 0.06	Lattice simulations	2008
0.99	RG study	2012
0.92	SD (LO, dynamic RPA)	2009
0.9 ± 0.2	Lattice simulations	2012
0.833	RG study	2008

Outline

Conclusion

We have studied dynamical "chiral" (flavour) symmetry breaking in QED_3 with N four-component fermions solving the Schwinger-Dyson gap equation in the large-N limit (+ extension to $\mathrm{QED}_{4,3}$ via a new mapping)

- the LO and NLO in the $1/N$-expansion were computed exactly
- all calculations were carried out in an arbitrary non-local gauge
- Nash's resummation of the wave function renormalization at the level of the gap equation was implemented
- a complete suppression of the gauge dependence of N_c at NLO was proved using Gusynin's prescription [Gusynin & Pyatkovskiy '16]
  - exact NLO + resummation: $N_c = 2.85$ (in perfect agreement with [Gusynin & Pyatkovskiy '16])
  - increasing support for the stability of the critical point
  - suggests that DχSB should take place for integer values $N \leq 2$

Powerful fully gauge-invariant approach
that may be extended (if needed) to NNLO