Elliptic polylogarithms and Feynman integrals

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Workshop on Multi-loop Calculations: Methods and Applications

LAPTH - Sorbonne Université
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String Theory

EFTs

Conformal Bootstrap

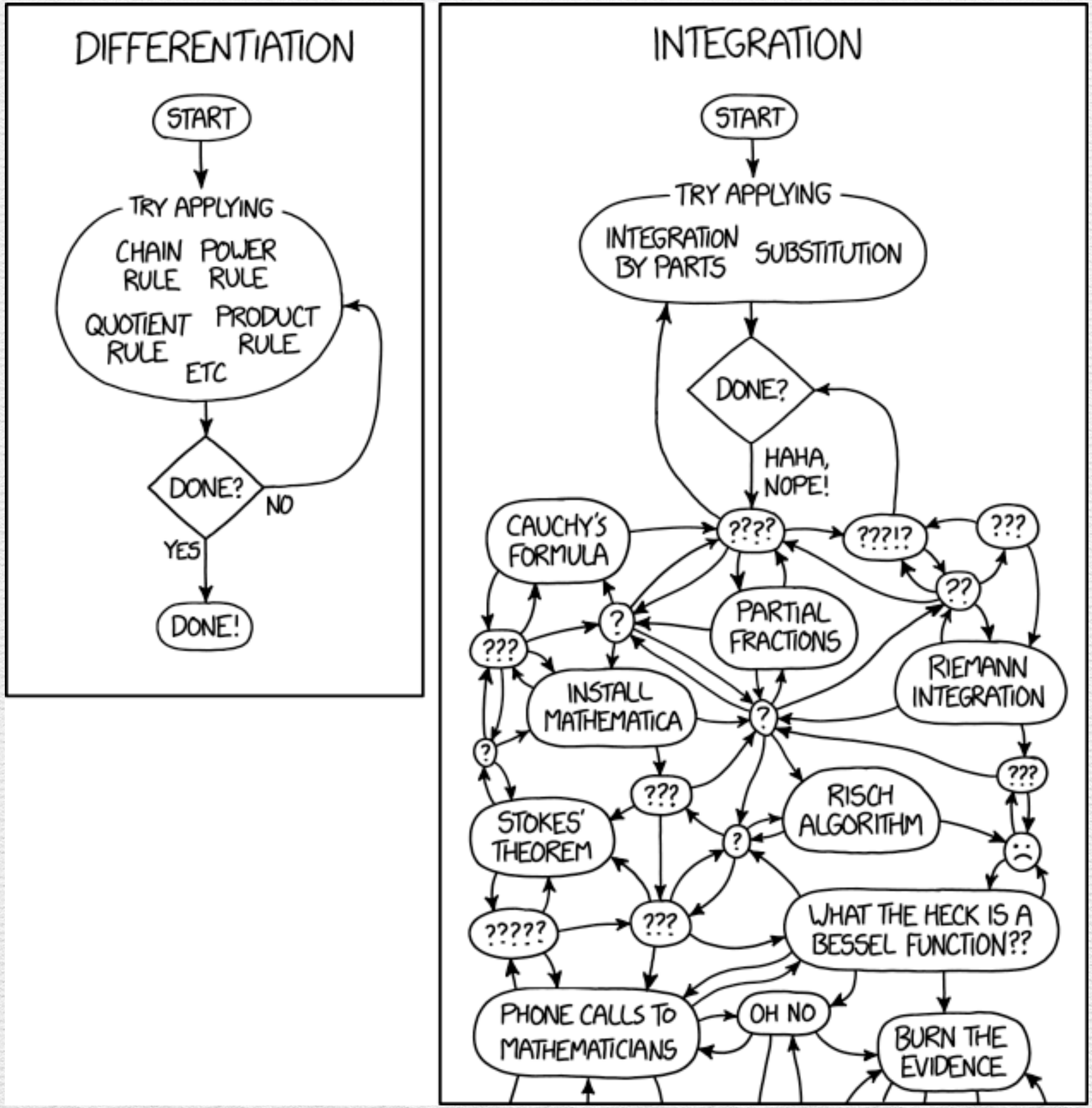
GR

Gravitational Physics

QFTs

Integrability

Particle Physics



[xkcd.com/2117]

String Theory

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Feynman Integrals

L-loop integral in D dimensions:

Mandelstam variables

tensor numerator

$$I_{\mu_1\cdots\mu_n}(\{s_{ij}\},\{m_i^2\}) = \int \prod_{i=1}^{L} \mathfrak{D}^D \ell_i \frac{T_{\mu_1\cdots\mu_n}}{D_1^{\nu_1}\cdots D_p^{\nu_p}}$$

propagators

$$(k_i^2 - m_i^2)$$

Feynman Integrals

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propagators

$(k_i^2 - m_i^2)$

Standard QFT material, just integrate it.

Integrating Feynman integrals requires a lot of creativity:

- UV/IR divergences — dimensional regularisation: $D = D_{\text{int}} - 2\,\epsilon$
- Feynman parameters $\displaystyle \frac{1}{D_1^{\nu_1}\cdots D_p^{\nu_p}} = \frac{\Gamma(\nu_1+\cdots+\nu_p)}{\Gamma(\nu_1)\cdots\Gamma(\nu_p)}\int_0^1 d\alpha_1\ldots d\alpha_p \frac{\delta(1-\sum_{i=1}^p\alpha_i)\prod_{i=1}^p\alpha_i^{\nu_i-1}}{(\sum_{i=1}^p\alpha_i D_i)^{\sum_{i=1}^p\nu_i}}$
- Integration-by-parts identities among integrals

$$\int d^d\ell\, v^\mu \frac{\partial}{\partial\ell^\mu}\frac{N}{D_1^{\nu_1}\,D_2^{\nu_2}\,\cdots D_n^{\nu_n}} = 0$$

- Independent integrals are called master integrals
- Differential equations for master integrals (also in canonical form)

— Kotikov '91 / Remiddi '97 / Gehrmann, Remiddi '99 / Henn '13 —

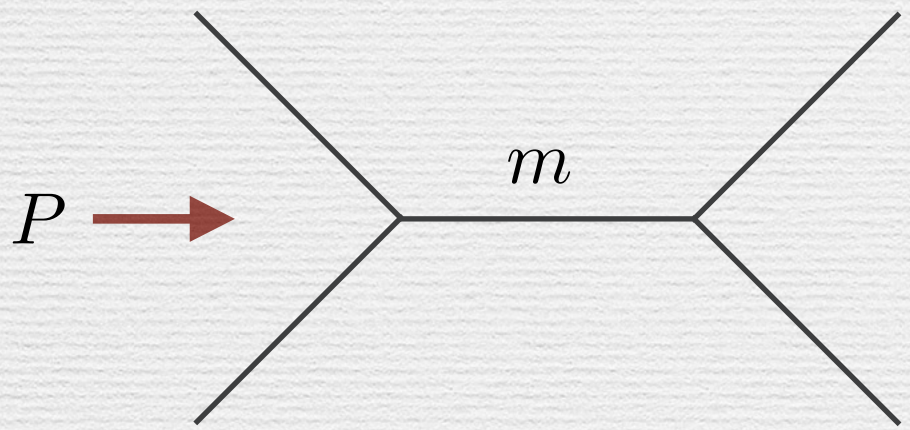
$dF = \epsilon\, dA\, F$ → Solution in terms of iterated integrals

→ Matrix of differential forms

It helps if you understand properties of the result:

- They evaluate to “special functions” which contain physical information in their analytic structure

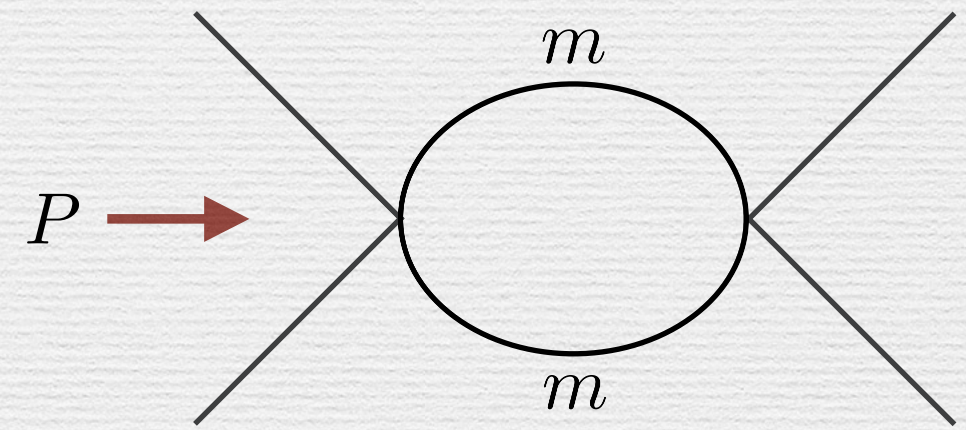
Tree level:



Pole at:

$$P^2 = m^2$$

Loop level:



Threshold starting at:

$$P^2 = (2m)^2$$

$\longrightarrow$ discontinuity

- The branch cut structure of loop integrals becomes ever more intricate with more legs, loops, scales

- Useful to study general functions that result from FIs

- Multivalued functions: $\log(e^{2\pi i}z) = \log(z) + 2\pi i$

- Most well studied case: Multiple Polylogarithms (MPLs)

  (all 1-loop examples and most 2-loop examples without internal masses)

In this talk we want to go beyond MPLs, but first let's understand what properties we would like to generalise

Multiple Polylogarithms (MPLs)

Logarithm

$$\log(z) = \int_1^z \frac{dt}{t}$$

Classical Polylogarithms

$$\mathrm{Li}_1(z) = -\log(1-z) \qquad \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t}\mathrm{Li}_{n-1}(t), \quad n > 1$$

Multiple Polylogarithms

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t) \quad a_i \in \mathbb{C}$$

$$\mathrm{Li}_n(z) = G(0,\ldots,0,1;z)$$

$$G(\underbrace{0,\ldots,0}_{n \text{ times}};z) = \frac{1}{n!}\log^n z$$

MPLs are iterated integrals of rational functions defined on a Riemann sphere with punctures $\longrightarrow$ integration kernels: $\dfrac{1}{t-a}$

- MPLs: Weight = Length = number of integrations

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t) \qquad a_i \in \mathbb{C}$$

$$G(\underbrace{0, \ldots, 0}_{n \text{ times}}; z) = \frac{1}{n!} \log^n z \qquad G(; z) \equiv 1$$

Weight 1 $\qquad i\pi = \log(-1)$

Weight n $\qquad G(a_1, \ldots, a_n; z)$

$$\zeta_n = -G(\vec{0}_{n-1}, 1; 1)$$

Lots of nice properties:

Total differential: $$dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log \frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

Symbol: $$\mathcal{S}(G(a_1,\ldots,a_n;z)) = \sum_{i=1}^{n} \mathcal{S}(G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)) \otimes \frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

Length n $\longrightarrow$ n -fold tensor product

In depth understanding of these functions was key to obtaining analytic results for Feynman integrals or even complete amplitudes

- Taming analytical expressions, functional identities
- Symbol bootstrap with MPL ansatz in N=4 SYM

— Caron-Huot, Dixon, Drummond, Duhr, Harrington, Henn, McLeod, Papathanaseou, Pennington, Spradlin, von Hippel —

MPLs are *pure*

What do you mean “Pure”?

- Definition based on total differential — Henn ’13 —

\# of integrations

A pure function of weight n is a function whose total derivative can be expressed in terms of pure functions of weight n-1 (times algebraic one-forms)

algebraic

$$dG(a_1, \ldots, a_n; z) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z)\, d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}$$

weight n

weight $n-1$

What do you mean "Pure"?

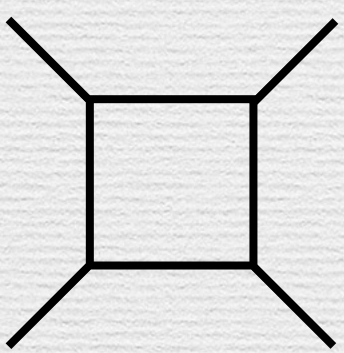
- Definition based on residues

— Arkani-Hamed, Bourjaily
Cachazo, Trnka '12 —

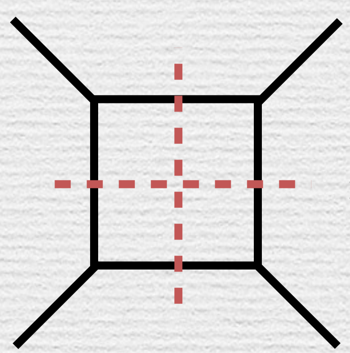
An integral is pure if all of its non-vanishing residues are the same up to a sign

"Integrals with unit leading singularity"

- Ex: massless box



$$= \frac{2}{st}\left[\frac{1}{\epsilon^2} - \frac{\log(st)}{\epsilon} + \log(-s)\log(-t) - \frac{2\pi^2}{3}\right]$$



$$= \pm\frac{1}{st}$$

(weight of ϵ is -1: $q^\epsilon = e^{\epsilon \log(q)}$)

What do you mean "Pure"?

- Definition based on residues

— Arkani-Hamed, Bourjaily
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An integral is pure if all of its non-vanishing residues are the same up to a sign

"Integrals with unit leading singularity"

Pure Feynman Integrals, when properly normalised:

- Are expressible in terms of pure functions
- Satisfy a differential equation system in canonical form

Pure integrals evaluate to pure functions

Differential equations in canonical form — Henn '13 —

Matrix of "dlog" forms

$$dF = \epsilon \, dA \, F$$

Vector of master integrals

For MPLs:
Natural solution in terms of
pure functions G as an expansion in ϵ

$$F = P\exp\left[\epsilon \int_\gamma dA\right] F_0$$

Pure integrals evaluate to pure functions

Differential equations in canonical form — Henn '13 —

Matrix of "dlog" forms

$$dF = \epsilon \, dA \, F$$

Vector of master integrals

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Natural solution in terms of
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$$F = P \exp\left[\epsilon \int_\gamma dA\right] F_0$$

What to do when the integral cannot
be evaluated in terms of MPLs?

How do we know an integral can be evaluated in terms of MPLs?

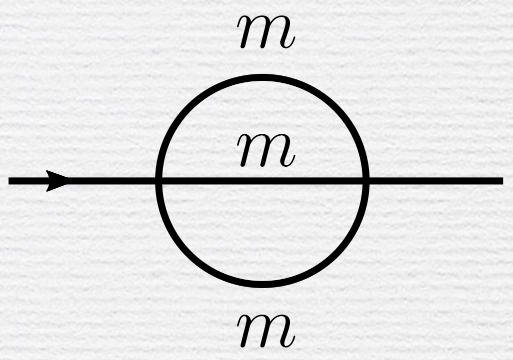
$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

At every integration step, the integrand can be written as an MPL times a rational function as above by using partial fractioning / change of variables

Concept of “linear reducibility” — Brown ‘08 / Panzer ‘15 —

This approach can be taken in a large variety of cases, but unfortunately not always

Ex: 2-loop massive sunrise in d=2



Two of the master integrals satisfy a coupled system of DE,
First master integral satisfies a 2nd order DE:

$$D\left(\frac{d^2}{da^2}, \frac{d}{da}\right) S_{111} = R(a) \qquad a = \frac{p^2}{m^2}$$

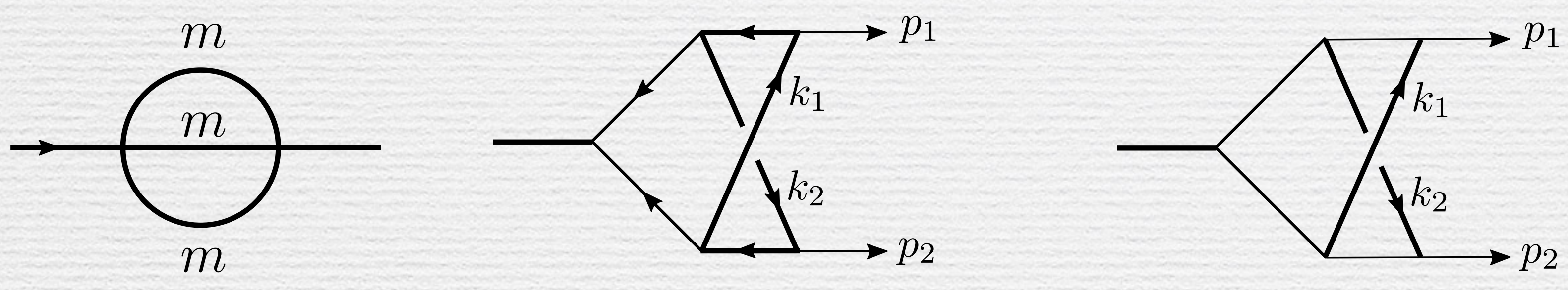
Homogeneous solution:

$$\mathrm{K}(\lambda) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda t^2)}}$$

Sqrt of quartic polynomial
— cannot be rationalised —

(complete elliptic integral of the 1st kind)

By now we know lots of examples that don't fit into the MPL framework:



and many more…

Goal: Develop a class of functions which is applicable in general for Feynman integrals of the elliptic kind (next-to-simplest):

Outline

- Elliptic generalisations of MPLs to functions on the elliptic curve w/ log singularities suitable for FI computations

- Generalise as many properties of MPLs to the elliptic case, in particular purity / uniform weight

Define *pure* elliptic MPLs (eMPLs)

- We seek to generalise the following to the elliptic case:

A function is called pure if it is unipotent and its total differential involves only pure functions and one-forms with at most logarithmic singularities.

(Unipotent: total diff has no homogeneous term)

Log singularities

$$dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log \frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

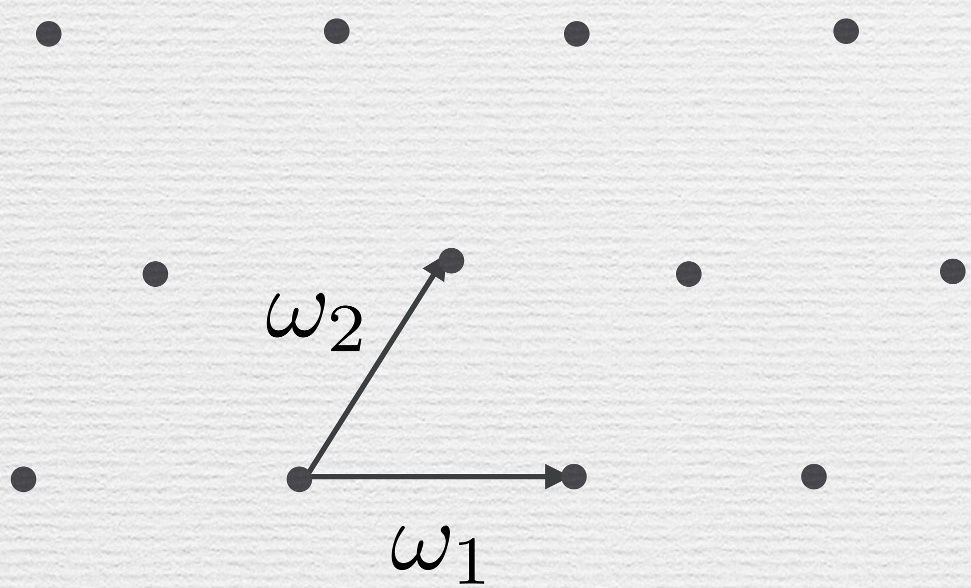
Pure Unipotent

Elliptic Polylogarithms on the torus

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

torus: $\mathbb{C}/\Lambda$

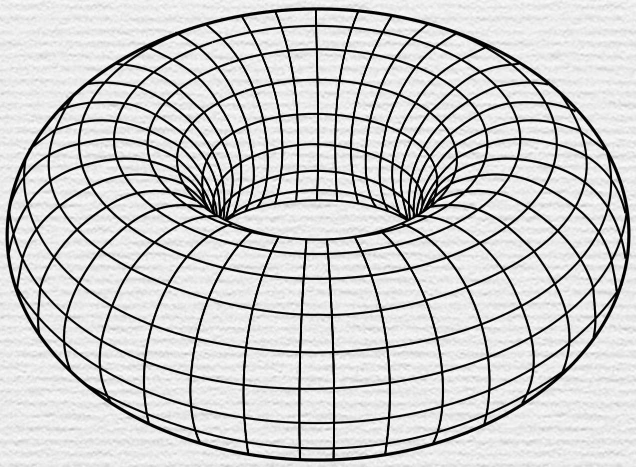
$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$



Modular group: $\mathrm{SL}(2,\mathbb{Z})$

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

$$\tau = \frac{\omega_2}{\omega_1} \to \frac{a\tau + b}{c\tau + d}$$



Elliptic Polylogarithms on the torus

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right)$$

$n_i \in \mathbb{N}$

$z_i \in \mathbb{C} \cup \{\infty\}$

Kernels defined through generating function:

$$F(z, \alpha, \tau) = \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z, \tau)\alpha^n = \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)}$$

Odd Jacobi theta function

Kernels have at most simple poles at lattice points

$\tilde{\Gamma}$ form a basis for all eMPLs

Elliptic Polylogarithms on the torus

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right) \qquad \begin{array}{l} n_i \in \mathbb{N} \\ z_i \in \mathbb{C} \cup \{\infty\} \end{array}$$

Kernels defined through generating function:

$$F(z, \alpha, \tau) = \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z, \tau) \alpha^n = \frac{\theta_1'(0, \tau) \theta_1(z + \alpha, \tau)}{\theta_1(z, \tau) \theta_1(\alpha, \tau)}$$

Odd Jacobi theta function

Weierstrass zeta function

$$g^{(0)}(z, \tau) = 1$$

$$g^{(1)}(z, \tau) = \zeta(z) - 2\frac{\eta_1}{\omega_1} z$$

$g^{(n)}(z, \tau)$ polynomial of degree n in $g^{(1)}(z, \tau)$

Like MPLs, $\tilde{\Gamma}$ satisfy nice properties

Total differential without homogeneous term (= unipotent)

— Broedel, Duhr, Dulat, Penante, Tancredi, 2018 —

$$d\widetilde{\Gamma}\left(A_1 \cdots A_k; z, \tau\right) = \sum_{p=1}^{k-1} (-1)^{n_{p+1}} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, {}^{0}_{0} \, A_{p+2} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p + n_{p+1})}$$

$$+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \left[\binom{n_{p-1}+r-1}{n_{p-1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}^{[r]} \, \hat{A}_p \, A_{p+1} \cdots A_k; z, \tau\right) \omega_{p,p-1}^{(n_p - r)} \right.$$

$$\left. - \binom{n_{p+1}+r-1}{n_{p+1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, \hat{A}_p \, A_{p+1}^{[r]} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p - r)} \right]$$

one-forms w/ log singularities

$$A_i^{[r]} \equiv \left({}^{n_i + r}_{z_i} \right) \qquad A_i^{[0]} \equiv A_i$$

$$\omega_{ij}^{(n)} = (dz_j - dz_i)\, g^{(n)}(z_j - z_i, \tau) + \frac{n\, d\tau}{2\pi i}\, g^{(n+1)}(z_j - z_i, \tau)$$

Important: $g^{(n)}(z, \tau)$ have at most simple poles for $z = m + n\tau,\ m, n \in \mathbb{Z}$

Like MPLs, $\tilde{\Gamma}$ satisfy nice properties

Total differential without homogeneous term (= unipotent)

— Broedel, Duhr, Dulat, Penante, Tancredi, 2018 —

$$
\begin{aligned}
d\widetilde{\Gamma}\left(A_1 \cdots A_k; z, \tau\right) = &\sum_{p=1}^{k-1} (-1)^{n_{p+1}} \,\widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, {}^{0}_{0} \, A_{p+2} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p+n_{p+1})} \\
&+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \left[\binom{n_{p-1}+r-1}{n_{p-1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}^{[r]} \, \hat{A}_p \, A_{p+1} \cdots A_k; z, \tau\right) \omega_{p,p-1}^{(n_p-r)} \right. \\
&\left. - \binom{n_{p+1}+r-1}{n_{p+1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, \hat{A}_p \, A_{p+1}^{[r]} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p-r)} \right]
\end{aligned}
$$

one-forms w/ log singularities

A function is called pure if it is unipotent and it has at most logarithmic singularities.

$\tilde{\Gamma}$ are pure!

So, we can use as guiding principle

An elliptic Feynman integral is pure if it is pure when expressed in terms of $\tilde{\Gamma}$

=

Linear combination of $\tilde{\Gamma}$ *with coefficients being rational numbers*

$$d(f(z,\tau)\tilde{\Gamma}(\ldots,z,\tau)) = (df(z,\tau))\tilde{\Gamma}(\ldots,z,\tau) + f(z,\tau)d(\tilde{\Gamma}(\ldots,z,\tau))$$

Homogeneous

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An elliptic Feynman integral is pure if it is pure when expressed in terms of $\tilde{\Gamma}$

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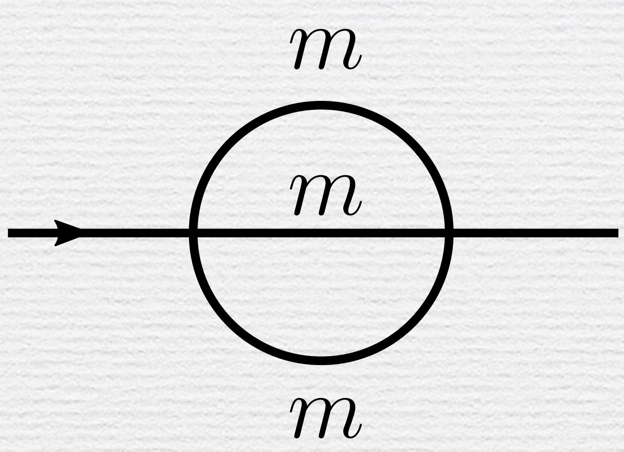
$$d(f(z,\tau)\tilde{\Gamma}(\ldots,z,\tau)) = (df(z,\tau))\tilde{\Gamma}(\ldots,z,\tau) + f(z,\tau)d(\tilde{\Gamma}(\ldots,z,\tau))$$

Homogeneous

Why bother defining another version of eMPLs?

Back to the sunrise, the maximal cut didn't look very much like $g^{(n)}(z,\tau)$ ...



$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$$

Elliptic curves

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4) \equiv P_4(x)$$

Vector of branch points of y : $\vec{a} = (a_1, a_2, a_3, a_4)$

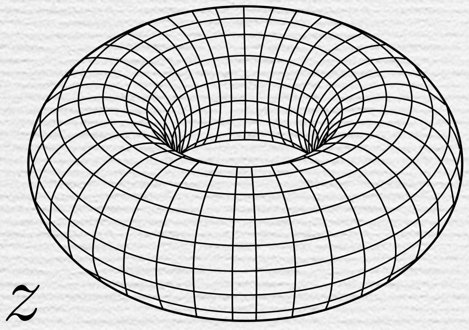
Periods:

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\,\mathrm{K}(\lambda) \qquad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)$$

$$\lambda = \frac{(a_1-a_4)(a_2-a_3)}{(a_1-a_3)(a_2-a_4)} \qquad c_4 = \frac{1}{2}\sqrt{(a_1-a_3)(a_2-a_4)}$$

We want: iterated integral of rational functions of x and y where $y^2 = P_4(x)$.

Elliptic Curves and Tori



$$\tau = \frac{\omega_2}{\omega_1}$$

vs. $$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \equiv P_4(x)$$

Kappa function $\kappa(.,\vec{a}) : \mathbb{C}/\Lambda_\tau \to \mathbb{C}$

$$(c_4 \kappa'(z))^2 = (\kappa(z) - a_1)(\kappa(z) - a_2)(\kappa(z) - a_3)(\kappa(z) - a_4)$$

$$y^2 = P_4(x)$$

$$(x, y) = (\kappa(z), c_4 \kappa'(z))$$

Abel's map

$$(x, y) \mapsto z_x \equiv \frac{c_4}{\omega_1} \int_{a_1}^{x} \frac{dx}{y} \mod \Lambda$$

To summarise:

We define a basis of eMPLs on the elliptic curve such that

1. They form a basis for all eMPLs
2. They are pure
3. They have definite parity $(x, y) \to (x, -y) \iff z \to -z$
4. They manifestly contain ordinary MPLs

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$n_i \in \mathbb{Z}$ is a label

$c_i \in \mathbb{C}$ indicate punctures (for $|n_i| = 1$)

Infinitely many kernels, Ψ_n but only $|n| \leq 2$ typically appear in explicit problems

Ex: $\Psi_0(0, x, \vec{a}) = \dfrac{c_4}{\omega_1\, y}$ $\qquad \Psi_1(c, x, \vec{a}) = \dfrac{1}{x - c}$

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix}; t, \vec{a})$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) &= dz_x \Big[g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad - \delta_{\pm n, 1} \Big(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \Big) \Big] \end{aligned}$$

Recall: $g^{(i)}(z, \tau)$ are the kernels of the eMPLs $\tilde{\Gamma}$

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$dx\, \Psi_{\pm n}(c, x, \vec{a}) = dz_x \Big[g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) - \delta_{\pm n, 1} \left(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big]$$

1. They form a basis for all eMPLs ✔

(one-to-one correspondence with basis of $\tilde{\Gamma}$)

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) \; &= dz_x \Big[g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad - \delta_{\pm n,1} \left(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

2. They are pure ✔

(Linear combination of $\tilde{\Gamma}$ with numeric coefficients)

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a})$$

$$n_i \in \mathbb{Z}$$

$$dx\, \Psi_{\boxed{\pm n}}(c, x, \vec{a}) = dz_x \Big[g^{(n)}(z_x - z_c, \tau) \boxed{\pm} g^{(n)}(z_x + z_c, \tau)$$
$$- \delta_{\pm n, 1} \left(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big]$$

3. They have definite parity ✔

(Recall $g^{(n)}(-z, \tau) = (-1)^n g^{(n)}(z, \tau)$)

Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\,\Psi_{\pm n}(c, x, \vec{a}) &= dz_x \Big[g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad \boxed{- \delta_{\pm n, 1} \Big(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \Big) \Big]} \end{aligned}$$

$$dx\,\Psi_1(c, x, \vec{a}) = \frac{dx}{x - c}, \qquad c \neq \infty$$

4. They manifestly contain ordinary MPLs

Recall from ordinary polylogs:

		Weight	Length
$i\pi$	$\longrightarrow$	1	0
ζ_k	$\longrightarrow$	k	0
$G(c_1, \ldots, c_k; x)$	$\longrightarrow$	k	k

Recall from ordinary polylogs:

		Weight	Length
$i\pi$	$\longrightarrow$	1	0
ζ_k	$\longrightarrow$	k	0
$G(c_1,\ldots,c_k;x)$	$\longrightarrow$	k	k

Empirically, by requiring relations between uniform weight functions, we postulate:

$\omega_1,\, \omega_2$	$\longrightarrow$	1	0
$\tau = \dfrac{\omega_2}{\omega_1}$	$\longrightarrow$	0	1
$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ z_1 & \ldots & z_k \end{smallmatrix}; z, \tau\right)$	$\longrightarrow$	$\sum_i n_i$	k
$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}\right)$	$\longrightarrow$	$\sum_i \lvert n_i \rvert$	k

We'll see in applications that using these definitions, results are of uniform weight

How to use this framework — step by step

1. Start from Feynman parametric integral

2. Do as many integrals as possible in terms of MPLs G

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

3. Reach a representation of the type $I = \int_0^1 \frac{dx}{y} \times (\text{bunch of Gs})$

4. Rewrite (bunch of Gs) as $\Psi_n(\ldots, x, \vec{a})\mathcal{E}_4(\ldots; x, \vec{a})$

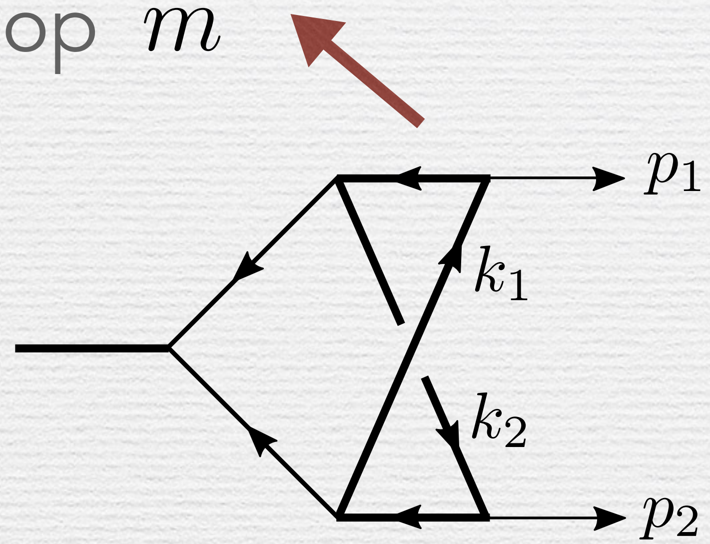
5. Integrate in terms of eMPLs

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix}; t, \vec{a})$$

Ex: $t\bar{t}$ production

— Tancredi, von Manteuffel '17 —

Massive loop m



$$a = m^2/S$$

$$I_{a_1,\ldots,a_7} = -\frac{1}{\pi^D}\int d^D k_1 d^D k_2 \frac{D_7^{-a_7}}{\prod_{i=1}^6 D_i^{a_i}}$$

$$p_1^2 = p_2^2 = 0$$

$$S = -2(p_1 \cdot p_2)$$

$$\begin{aligned} D_1 &= k_1^2 - m^2, \quad & D_3 &= (k_1 - p_1)^2 - m^2, \quad & D_5 &= (k_1 - k_2 - p_1)^2, \\ D_2 &= k_2^2 - m^2, \quad & D_4 &= (k_2 - p_2)^2 - m^2, \quad & D_6 &= (k_2 - k_1 - p_2)^2, \quad D_7 = k_1 \cdot p_2. \end{aligned}$$

Two master integrals: $M_1 = I_{1,1,1,1,1,1,0}\,, \quad M_2 = I_{2,1,1,1,1,1,0}$

First master $M_1 = I_{1,1,1,1,1,1,0}$

After doing all but one integration following the given steps:

$$
\begin{aligned}
M_1 &= \frac{2a^2}{3}\int_0^1 \frac{d\bar{x}_2}{y} \\
&\times \Bigg[6\Bigg(G\left(\left(\bar{x}_2-1\right)\bar{x}_2;a\right)\left(G_-\left(-\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+2G_-\left(\bar{x}_2;\bar{x}_2\right)\right) \\
&+G\left(0;\bar{x}_2\right)\left(2G_-\left(\bar{x}_2;\bar{x}_2\right)-G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)\right)-G\left(1;\bar{x}_2\right)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right) \\
&+2G_-\left(0,\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+G_-\left(-\frac{a}{\bar{x}_2-1},\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right) \\
&+2G_-\left(\bar{x}_2,\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right)+G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\frac{a}{\bar{x}_2-1};\bar{x}_2\right) \\
&-2\log(a)G_-\left(\bar{x}_2;\bar{x}_2\right)+\log(a)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)+2G\left(1;\bar{x}_2\right)G_-\left(\bar{x}_2;\bar{x}_2\right)\Bigg) \\
&-G_-\left(\bar{x}_2\right)\Big(6G\left(0;\bar{x}_2\right)G\left(\left(1-\bar{x}_2\right)\bar{x}_2;a\right)+6G\left(1;\bar{x}_2\right)G\left(\left(1-\bar{x}_2\right)\bar{x}_2;a\right) \\
&+6G\left(0,\left(1-\bar{x}_2\right)\bar{x}_2;a\right)+6G\left(0,\left(\bar{x}_2-1\right)\bar{x}_2;a\right)-6\log(a)G\left(\left(1-\bar{x}_2\right)\bar{x}_2;a\right)+\pi^2\Big)\Bigg],
\end{aligned}
$$

First master $M_1 = I_{1,1,1,1,1,1,0}$

After doing all but one integration following the given steps:

$$
\begin{aligned}
M_1 = \frac{2a^2}{3}\int_0^1 \frac{d\bar{x}_2}{y} \\
\times \Bigg[6\Bigg(& G\left((\bar{x}_2-1)\,\bar{x}_2;a\right)\left(G_-\left(-\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+2G_-\left(\bar{x}_2;\bar{x}_2\right)\right) \\
&+G\left(0;\bar{x}_2\right)\left(2G_-\left(\bar{x}_2;\bar{x}_2\right)-G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)\right)-G\left(1;\bar{x}_2\right)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right) \\
&+2G_-\left(0,\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+G_-\left(-\frac{a}{\bar{x}_2-1},\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right) \\
&+2G_-\left(\bar{x}_2,\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right)+G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\frac{a}{\bar{x}_2-1};\bar{x}_2\right) \\
&-2\log(a)G_-\left(\bar{x}_2;\bar{x}_2\right)+\log(a)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)+2G\left(1;\bar{x}_2\right)G_-\left(\bar{x}_2;\bar{x}_2\right)\Bigg) \\
&-G_-\left(\bar{x}_2\right)\Bigg(6G\left(0;\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right)+6G\left(1;\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right) \\
&+6G\left(0,(1-\bar{x}_2)\,\bar{x}_2;a\right)+6G\left(0,(\bar{x}_2-1)\,\bar{x}_2;a\right)-6\log(a)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right)+\pi^2\Bigg)\Bigg],
\end{aligned}
$$

Bunch of Gs

First master $M_1 = I_{1,1,1,1,1,1,0}$

After doing all but one integration following the given steps:

$$M_1 = \frac{2a^2}{3}\int_0^1 \frac{d\bar{x}_2}{y}$$

$$y^2 = P_4(\bar{x}_2) = \bar{x}_2(\bar{x}_2-1)(\bar{x}_2-b_+)(\bar{x}_2-b_-)$$
$$\vec{b} = (b_-,\,1,\,0,\,b_+)\,, \quad b_\pm = \frac{1}{2}(1\pm\sqrt{1-16a})$$

Elliptic curve

$$\times\Bigg[6\Bigg(G\left((\bar{x}_2-1)\,\bar{x}_2;a\right)\left(G_-\left(-\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+2G_-\left(\bar{x}_2;\bar{x}_2\right)\right)$$
$$+G\left(0;\bar{x}_2\right)\left(2G_-\left(\bar{x}_2;\bar{x}_2\right)-G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)\right)-G\left(1;\bar{x}_2\right)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)$$
$$+2G_-\left(0,\frac{a}{\bar{x}_2-1};\bar{x}_2\right)+G_-\left(-\frac{a}{\bar{x}_2-1},\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right)$$
$$+2G_-\left(\bar{x}_2,\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2};\bar{x}_2\right)+G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\frac{a}{\bar{x}_2-1};\bar{x}_2\right)$$
$$-2\log(a)G_-\left(\bar{x}_2;\bar{x}_2\right)+\log(a)G_-\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2;\bar{x}_2\right)+2G\left(1;\bar{x}_2\right)G_-\left(\bar{x}_2;\bar{x}_2\right)\Bigg)$$
$$-G_-\left(\bar{x}_2\right)\Bigg(6G\left(0;\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right)+6G\left(1;\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right)$$
$$+6G\left(0,(1-\bar{x}_2)\,\bar{x}_2;a\right)+6G\left(0,(\bar{x}_2-1)\,\bar{x}_2;a\right)-6\log(a)G\left((1-\bar{x}_2)\,\bar{x}_2;a\right)+\pi^2\Bigg)\Bigg]\,,$$

Bunch of Gs

First master

$$M_1 = \Omega_1^{(t\bar{t})} \tilde{M}_1$$

$$\Omega_1^{(t\bar{t})} = -\frac{16\, a^2\, \omega_1}{m^4(1-\sqrt{1-16a})} \qquad \tilde{M}_1 = 5\, T_{1+}(a) + 3\, T_{1-}(a) + \mathcal{O}(\epsilon)$$

$$T_{1+}(a) = \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1-r_+ \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_+ \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1-r_+ \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_+ \end{smallmatrix};1\right) \;,$$

$$T_{1-}(a) = -\,\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 1 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 1 \end{smallmatrix};1\right)$$
$$+ \log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1-r_- \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & r_- \end{smallmatrix};1\right)\right]$$

$$r_\pm = \frac{1}{2}(1-\sqrt{1\pm 4a})$$

$$a = m^2/(-q^2)$$

First master

$$M_1 = \Omega_1^{(t\bar{t})} \tilde{M}_1$$

$$\Omega_1^{(t\bar{t})} = -\frac{16\, a^2\, \omega_1}{m^4(1-\sqrt{1-16a})} \qquad \tilde{M}_1 = 5\, T_{1+}(a) + 3\, T_{1-}(a) + \mathcal{O}(\epsilon)$$

$$T_{1+}(a) = \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1-r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1-r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_+ \end{smallmatrix}; 1\right),$$

$$T_{1-}(a) = -\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 0 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 1 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 0 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 1 \end{smallmatrix}; 1\right)$$

$$+ \log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1-r_- \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\cdots\right)\right]$$

$$r_\pm = \frac{1}{2}(1-\sqrt{1 \pm 4a})$$

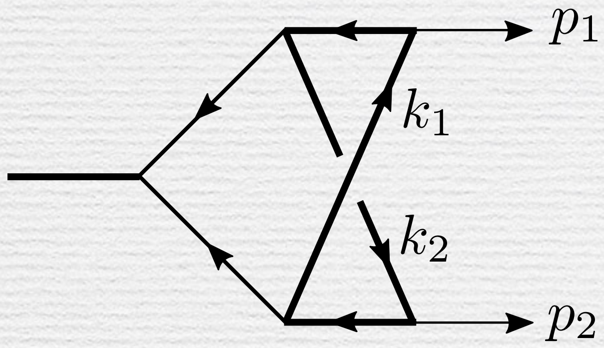
$$a = m^2/(-q^2)$$

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x\right) \to \sum_i |n_i|$$

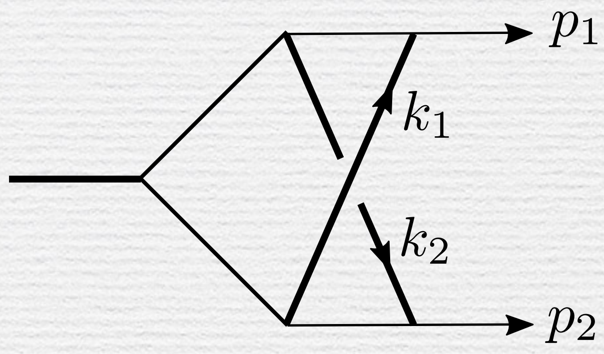
$$\omega_1, \pi \to 1$$

Manifestly pure of weight 4!

If we carry on:



— Tancredi, von Manteuffel '17 —



— Aglietti, Bonciani '07 —

2 master integrals

3 master integrals

- In both cases, possible to change basis to a basis of uniform weight!
- Unclear a priori
- Can it be done in general?
- What benefits can this bring?

Conclusions

- Pure elliptic polylogarithms seem to capture interesting structure of certain Feynman integrals

- General class of functions which deserves to be studied in depth if we seek manageable analytical expressions

- Efficient numerical evaluation to the same level of MPLs one day? [Weinzierl's talk]

- Not enough for integrals with more complicated geometry (K3 surfaces, CYs, multiple elliptic curves)

- Hopefully this formalism is useful outside of the realm of Feynman integrals too — curious to see room for potential applications in this workshop!

Back-up: Explicit eMPLs integration kernels

$$\Psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1\, y}$$

$$\Psi_1(c,x,\vec{a}) = \frac{1}{x-c}\,,$$

$$\Psi_{-1}(c,x,\vec{a}) = \frac{y_c}{y(x-c)} + Z_4(c,\vec{a})\,\frac{c_4}{y}\,,$$

$$\Psi_1(\infty,x,\vec{a}) = -Z_4(x,\vec{a})\,\frac{c_4}{y}\,,$$

$$\Psi_{-1}(\infty,x,\vec{a}) = \frac{x}{y} - \frac{1}{y}\left[a_1 + 2c_4\, G_*(\vec{a})\right]$$

$$G_*(\vec{a}) \equiv \frac{1}{\omega_1}\, g^{(1)}(z_*,\tau)$$

Image of ∞ on the torus

$$y_c = \sqrt{P_4(c)}$$

Requirement of simple poles at most introduces new building block:

Double pole at $x=\infty$

$$Z_4(x) \equiv \int_{a_1}^{x} dx'\, \Phi_4(x')$$

Simple pole at $x=\infty$