Multi-loop Calculations:

Methods and Applications

Elliptic multiple zeta values

and modular forms in string amplitudes

Oliver Schlotterer (Uppsala University)

based on work 2014 – 2018 in collaboration with J. Broedel, J. Gerken,

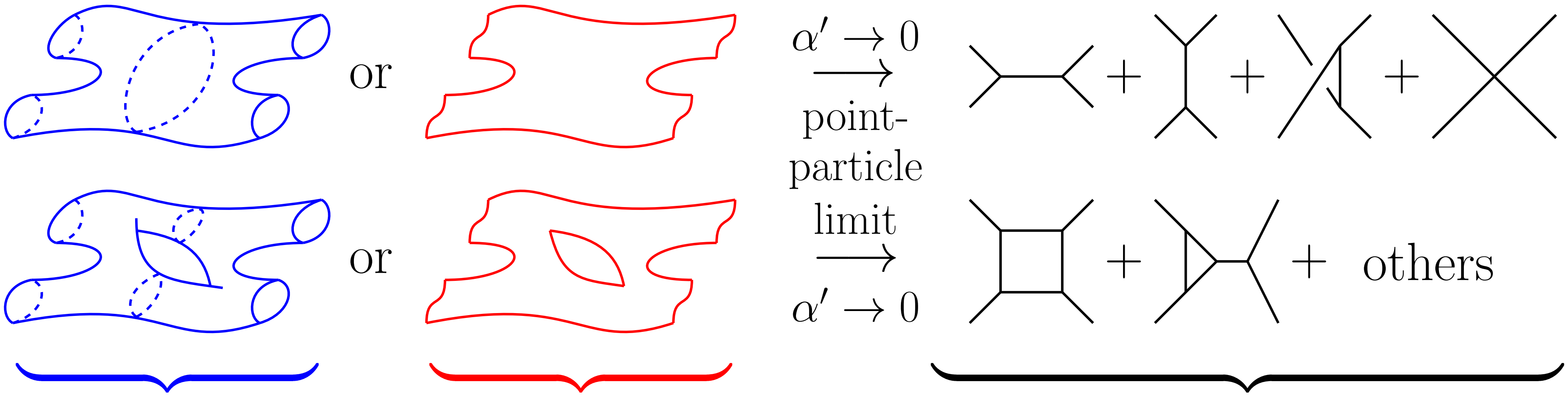
A. Kleinschmidt, C. Mafra, N. Matthes, O. Schnetz, F. Zerbini

15.05.2019

Intro I – string perturbation theory

String amplitudes $\longleftrightarrow$ Riemann surfaces as "fattened" Feynman diag's

loop order in perturbation theory = genus of the Riemann surface



closed-string states:
external gravitons

open-string states:
non-abelian gauge bosons

convenient organization of loop integrand
"gravity = (gauge theory)^2" (BCJ)

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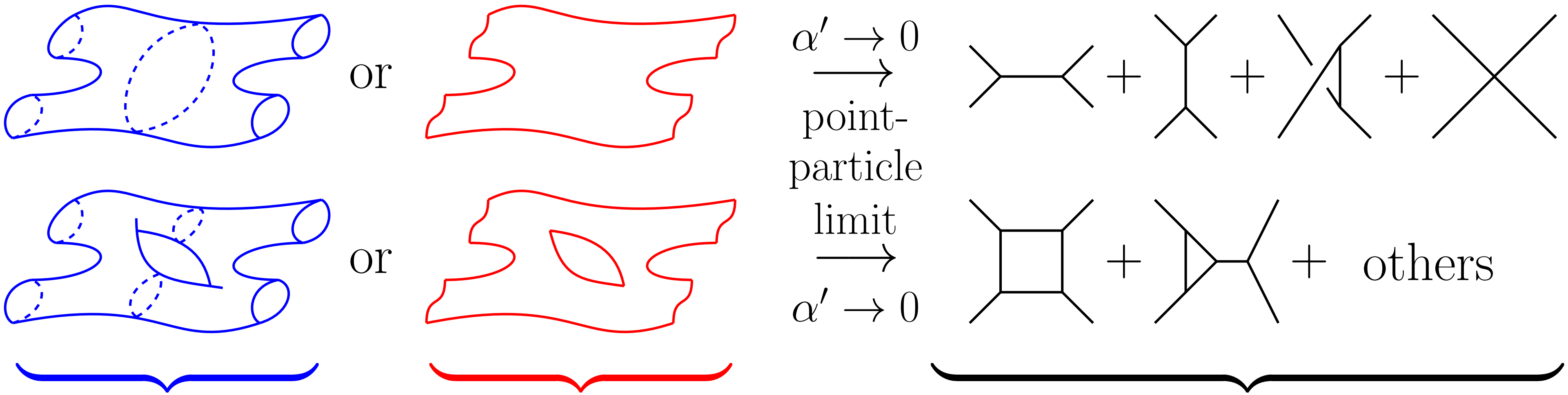


closed-string states: external gravitons

open-string states: non-abelian gauge bosons

convenient organization of loop integrand "gravity = (gauge theory)^2" (BCJ)

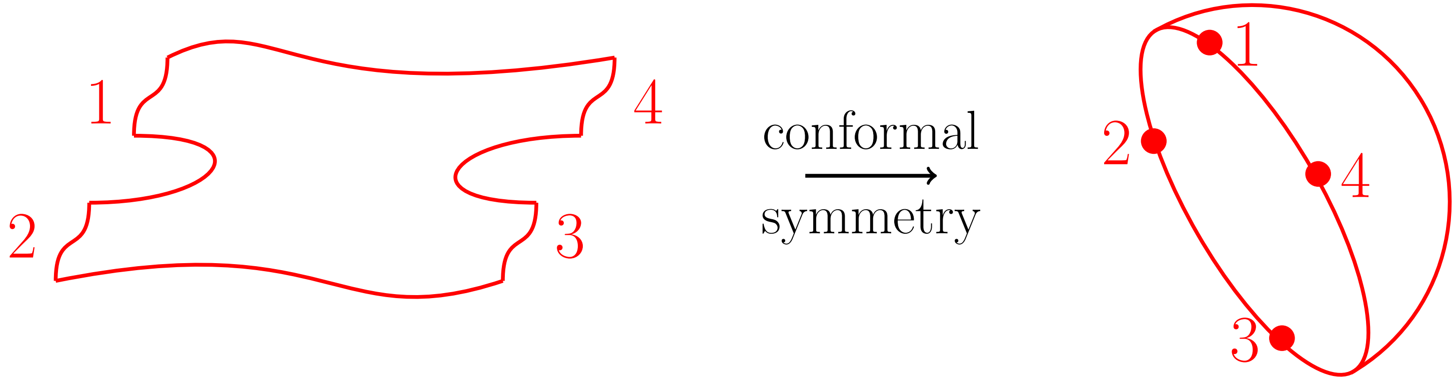
This talk: Study corrections to field theory $\sim$ inverse string tension α'

$\Longrightarrow$ rewarding laboratory for iterated integrals, multiple zeta values, polylogarithms, elliptic generalizations & modular forms

Intro I – string perturbation theory

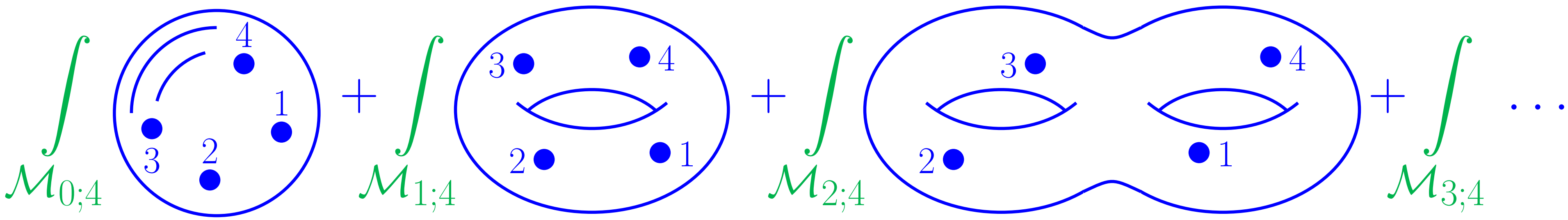
Map external states to punctures • on the Riemann surface, e.g.

open strings
at tree level:



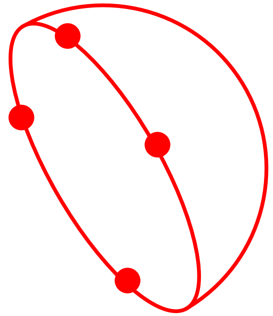
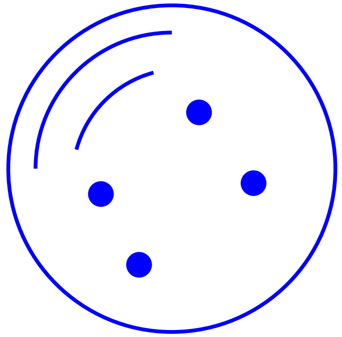
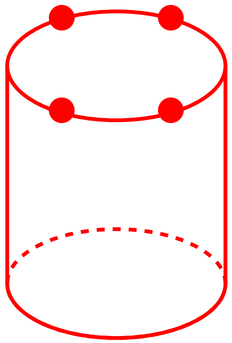
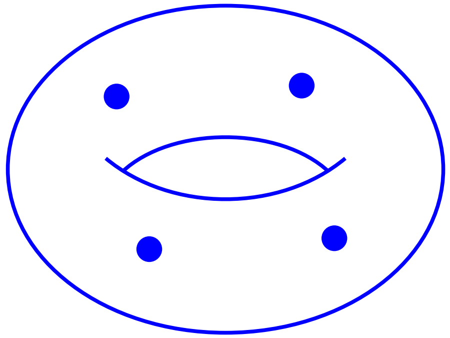
String amplitudes (n points, g loop) $\leftrightarrow$ integrals over moduli spaces $\mathcal{M}_{g;n}$

of n-punctured Riemann surfaces of genus g (with / without boundary),



α'-expansions $\leftrightarrow$ generating series for (large classes of) periods of $\mathcal{M}_{g;n}$.

Intro II – periods of moduli spaces at genus 0 & 1

	open strings		closed strings	
tree level	disk		sphere	
one loop	cylinder Möbius strip		torus	

Intro II – periods of moduli spaces at genus 0 & 1

	open strings	closed strings
tree level	disk $\Rightarrow$ multiple zeta values (MZVs) $=$ polylog's at $z=1$	sphere $\Rightarrow$ single-valued MZVs $=$ single-valued polylog's at $z=1$
one loop	cylinder / Möbius strip $\Rightarrow$ elliptic MZVs	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ $\Rightarrow$ modular graph forms (modular covariant fct's of τ)

Intro II – periods of moduli spaces at genus 0 & 1

	open strings	closed strings
tree level	disk $\Rightarrow$ multiple zeta values (MZVs) = polylog's at $z=1$	sphere $\Rightarrow$ single-valued MZVs = single-valued polylog's at $z=1$
one loop	cylinder / Möbius strip $\Rightarrow$ elliptic MZVs	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ $\Rightarrow$ modular graph forms (modular covariant fct's of τ)

Laboratory for multiple (elliptic) polylogarithms in Feynman integrals,

e.g. iterated τ-integrals over modular forms as a common theme

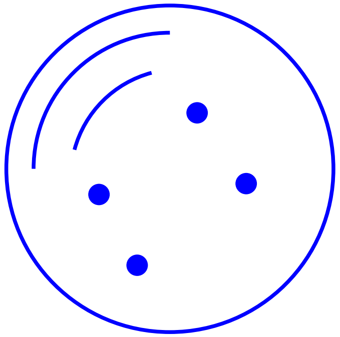
[see talks of Brenda Penante and Stefan Weinzierl]

Intro III – from open to closed strings

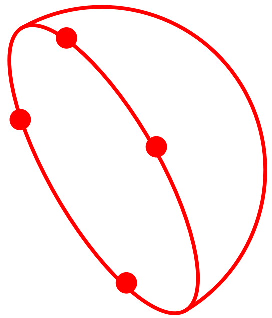
At tree level, can obtain closed-string α'-expansions

from single-valued projection "**sv**" of open-string α'-expansions

closed strings: sphere



$=$ **sv** $\Big\{$



open strings: disk $\Big\}$

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor, Vanhove, Zerbini]

$\rightarrow$ significant cleanup of KLT rel's (closed-string tree) = (open-string tree)^2

[Kawai, Lewellen, Tye 1986]

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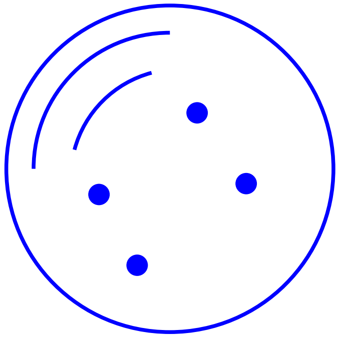

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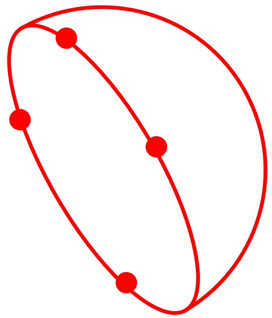


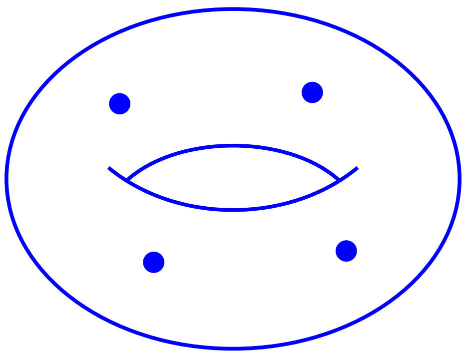
open strings: disk $\Bigg\}$

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor, Vanhove, Zerbini]

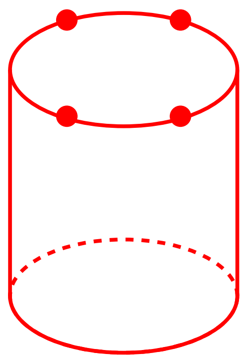
At one loop, propose examples of elliptic single-valued projection "**esv**"

by comparing closed-string α'-expansions $\leftrightarrow$ open-string α'-expansions

closed strings: torus



$\leftrightarrow$ **esv** $\Bigg\{$



open strings: cylinder $\Bigg\}$

[Brödel, Gerken, Kleinschmidt, OS, Zerbini]

Outline

I. Tree-level warmup

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor, Vanhove, Zerbini]

II. Elliptic MZVs and open strings at one loop

[Brödel, Mafra, Matthes, OS 1412.5535 & Brödel, Matthes, OS 1507.02254]

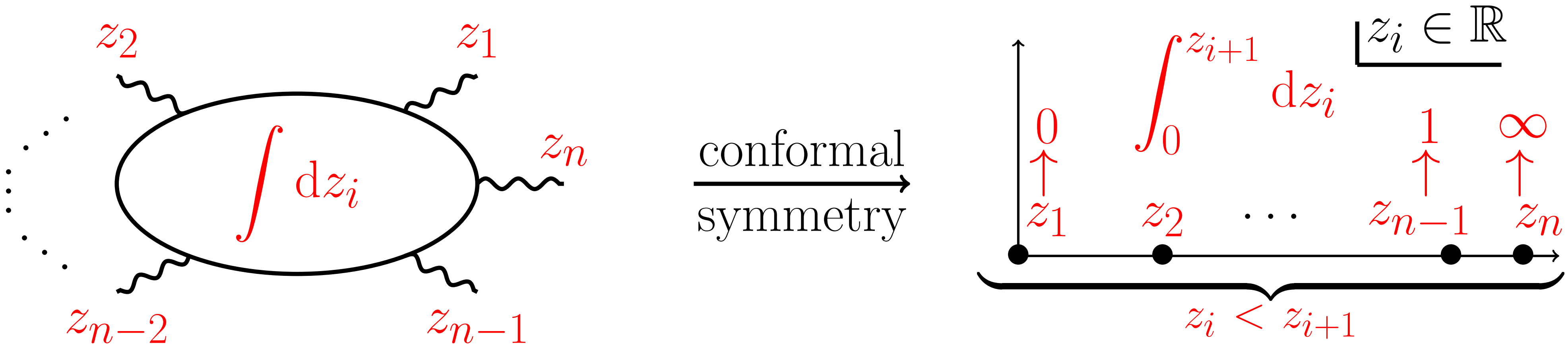
III. From closed strings to an elliptic single-valued map

[Brödel, OS, Zerbini 1803.00527 & Gerken, Kleinschmidt, OS 1811.02548]

IV. Conclusions & Outlook

I. Tree-level warmup

I. 1 Four open strings on the disk



Veneziano amplitude 1968 (4pt tree level, massless open-string states)

involving dim'less Mandestam invariants $s_{ij} := 2\alpha' k_i \cdot k_j$

$$
\begin{aligned}
Z_{\text{4-pt}} &= \int_{0=z_1}^{1=z_3} \frac{\mathrm{d}z_2}{z_2}\, z_2^{s_{12}} (1-z_2)^{s_{23}} = \frac{\Gamma(s_{12})\,\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\
&= \frac{1}{s_{12}} \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right) \\
&= \frac{1}{s_{12}} - \zeta_2\, s_{23} + \zeta_3\, s_{23}\,(s_{12}+s_{23}) + \ldots
\end{aligned}
$$

Expansion in α' or $s_{ij} \Rightarrow$ all Riemann zeta values $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$.

I. 2 Four closed strings on the sphere

Again fix $(z_1, z_3, z_4) \to (0, 1, \infty)$, integrate $z = z_2$

and use dim'less Mandestam invariants $s_{ij} := 2\alpha' k_i \cdot k_j$

$$J_{4\text{-pt}} = \frac{1}{\pi} \int_{\mathbb{C}\setminus\{0,1,\infty\}} \mathrm{d}^2 z \, \frac{|z|^{2s_{12}} \, |1-z|^{2s_{23}}}{z \, \bar{z} \, (1-\bar{z})} = \frac{1}{s_{12}} \prod_{i<j}^{3} \frac{\Gamma(1+s_{ij})}{\Gamma(1-s_{ij})}$$

$$= \frac{1}{s_{12}} \exp\left(-2 \sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1} \left[s_{12}^{2k+1} + s_{23}^{2k+1} + s_{13}^{2k+1} \right] \right)$$

Only ζ_{2k+1} at odd argument (no ζ_{2k} from open-string case)

$$Z_{4\text{-pt}} = \frac{1}{s_{12}} \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right)$$

Formally, at the level of α'-expansions, relate closed & open strings via

$$J_{4\text{-pt}} = Z_{4\text{-pt}} \Big|_{\substack{\zeta_{2k+1} \to 2\,\zeta_{2k+1} \\ \zeta_{2k} \to 0}}$$

I. 3 Single-valued MZVs

α'-expansion of n-point tree amplitudes involves multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \,, \qquad n_r \geq 2$$

Define single-valued projection **sv** of MZVs via their polylogarithm origin

[Schnetz 1302.6445 & Brown 1309.5309]

$$\begin{tikzcd}
\text{MZVs} \arrow[r, dashed, "\mathbf{sv}"'] & \text{single-valued MZVs} \\
\begin{array}{c}\text{polylogarithms}\\ G(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to 1"] \arrow[r, "\begin{array}{c}\text{Francis Brown 2004:}\\ \exists \text{ natural procedure to}\end{array}", "\text{cancel monodromies}"'] & \begin{array}{c}\text{single-valued polylog's}\\ G_{\rm sv}(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to 1"']
\end{tikzcd}$$

$$\text{e.g.} \quad G(1;z) \;=\; \log(1-z) \quad \longrightarrow \quad G_{\rm sv}(1;z) = \log|1-z|^2$$

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\end{tikzcd}$$

$$\mathbf{sv}(\zeta_{2k}) = 0\,, \qquad \mathbf{sv}(\zeta_{2k+1}) = 2\,\zeta_{2k+1}\,, \qquad \mathbf{sv}(\zeta_{3,5}) = -10\,\zeta_3\,\zeta_5\,, \qquad \text{etc.}$$

I. 4 Closed-string trees = sv(open-string trees)

At four points, $\mathbf{sv}(\zeta_{2k}) = 0$ and $\mathbf{sv}(\zeta_{2k+1}) = 2\,\zeta_{2k+1}$ underpins

$$\mathbf{sv}\,\underbrace{\left(\int_0^1 \frac{\mathrm{d}z}{z}\, z^{s_{12}}\,(1-z)^{s_{23}}\right)}_{\text{disk int. of } A^{\text{tree}}_{\text{open}}} = \underbrace{\frac{1}{\pi}\int_{\mathbb{C}\backslash\{0,1,\infty\}} \mathrm{d}^2 z\, \frac{|z|^{2s_{12}}\,|1-z|^{2s_{23}}}{z\,\bar{z}\,(1-\bar{z})}}_{\text{sphere int. of } M^{\text{tree}}_{\text{closed}}}$$

Same correspondence holds for n-point disk / sphere integrals

- conjectured after order-by-order inspection of α'-expansion
  [OS, Stieberger 1205.1516; Stieberger 1310.3259; Stieberger, Taylor 1401.1218]

- announced as a theorem
  [Brown: talk at String Math 2018 (Sendai, Japan); Brown, Dupont 1810.07682]

- physicist's proof (assuming e.g. standard transcendentality conjectures)
  [OS, Schnetz 1808.00713]

- alternative derivation of $M^{\text{tree}}_{\text{closed}} \in \mathbf{sv}(\text{MZV})$ via single-valued correlators
  [Vanhove, Zerbini 1812.03018]

II. Elliptic MZVs and open strings at one loop

II. 1 Four open strings on a cylinder

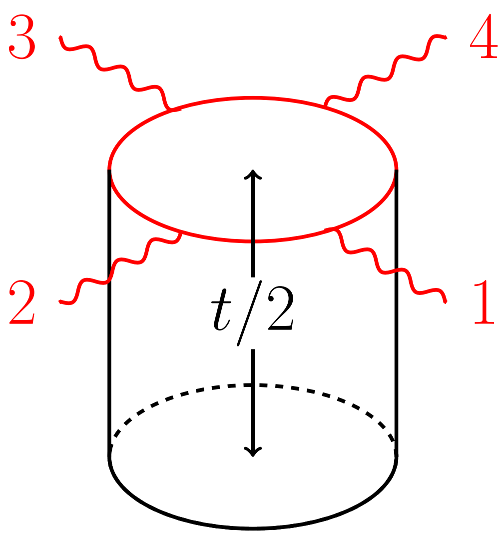
Cylinder contribution to planar one-loop amplitude $\sim \mathrm{Tr}(t^1 t^2 t^3 t^4)$

$$A^{\text{1-loop}}_{\text{open}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}t \; I_{1234}(s_{ij},\tau=it)$$

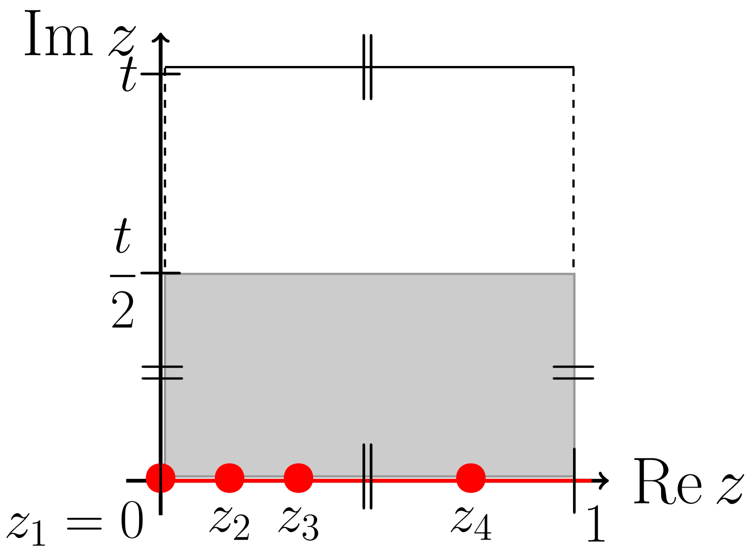
$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4 \;\exp\Big(\sum_{i<j}^4 s_{ij}P(z_i-z_j,\tau)\Big)$$

[Brink, Green, Schwarz 1982]

with $\sum_{i<j}^4 s_{ij} = 0$ and Green function $\partial_z P(z,\tau) = \partial_z \log\theta(z,\tau) + 2\pi i \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}$.



$\xrightarrow[\text{“half a torus”}]{\text{parametrized as}}$



II. 1 Four open strings on a cylinder

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4 \ \exp\Big(\sum_{i<j}^{4} s_{ij} \underbrace{P(z_i-z_j,\tau)}_{P_{ij}}\Big)$$

- Taylor expand $\exp(s_{ij}P_{ij}) = \sum_{n=0}^{\infty} \frac{1}{n!}(s_{ij}P_{ij})^n$ for each pair $1 \leq i < j \leq 4$
- integrating $\prod_{i<j}(P_{ij})^{n_{ij}}$ over cyl. boundary $\Rightarrow$ elliptic MZVs (eMZVs)

[Brödel, Mafra, Matthes, OS 1412.5535]

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  [Brödel, Mafra, Matthes, OS 1412.5535]
- eMZVs are proper subset of iterated τ-integrals $\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau)$

  over holomorphic Eisenstein series $\mathrm{G}_k(\tau)$ with $\mathbb{Q}[\mathrm{MZV},\frac{1}{2\pi i}]$ coeff's

  [Enriquez 1301.3042 & Brödel, Matthes, OS 1507.02254]
- cylinder $\leftrightarrow$ specialize $\tau = it$ with $t \in \mathbb{R}_+$, Moebius strip has $\tau = \frac{1}{2} + it$

II. 2 Iterated Eisenstein integrals

Holomorphic Eisenstein series ($k \geq 4$ even, $q = e^{2\pi i\tau}$) & $\mathrm{G}_0 = -1$

$$\mathrm{G}_k(\tau) \;=\; \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \;=\; 2\,\zeta_k \;+\; \frac{2\,(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}$$

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Define iterated Eisenstein integrals recursively by $\mathcal{E}_0(\,;\tau) = 1$ and

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} \mathrm{d}\tau' \, \mathrm{G}_{k_r}^0(\tau') \, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

$k_1 \geq 4 \;\Rightarrow$ convergent integrals by zero-mode subtraction $\mathrm{G}_{k_1}^0 = \mathrm{G}_{k_1} - 2\zeta_{k_1}$

$$\text{e.g.} \quad \mathcal{E}_0(k,\underbrace{0,0,\ldots,0}_{p-1};\tau) = \frac{-2}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{(mn)^p} q^{mn}$$

q-expansion straightforwardly inherited from above $\mathrm{G}_k^0(\tau)$.

II. 2 Iterated Eisenstein integrals

Back to open-string integral

$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4 \ \exp\Big(\sum_{i<j}^{4} s_{ij}\, P(z_i - z_j,\tau)\Big)$$

with Mandelstam relations $s_{34}=s_{12}, \ \ s_{14}=s_{23}$ & $s_{13}=s_{24}=-s_{12}-s_{23}$

$$\begin{aligned} I_{1234}(s_{ij},\tau) \ &= \ \frac{1}{6} \ + \ \frac{3s_{13}}{2\pi^2}\big[\zeta_3 \ - \ 6\,\mathcal{E}_0(4,0,0;\tau)\big] \\ &\quad + \frac{60}{\pi^2}\,(s_{13}^2+2s_{12}s_{23})\Big[\mathcal{E}_0(6,0,0,0;\tau) - \tfrac{\zeta_4}{120}\Big] \\ &\quad - 2\,(s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big[\mathcal{E}_0(4,0;\tau) - \tfrac{\zeta_2}{12}\Big] + \mathcal{O}(\alpha'^3) \end{aligned}$$

Order by order in s_{ij}, get eMZVs and therefore iterated Eisenstein integrals

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Order by order in s_{ij}, get eMZVs and therefore iterated Eisenstein integrals

... and the same is true for non-planar one-loop amplitudes $\sim \mathrm{Tr}(t^1t^2)\mathrm{Tr}(t^3t^4)$

[Broedel, Matthes, Richter, OS 1704.03449]

II. 3 Symmetrized open-string integral

To connect with closed strings, combine permutations of $\int_{0<z_2<z_3<z_4<1}$

$$I_{\substack{\text{symm}\\ \text{open}}}(s_{ij},\tau) \;=\; \sum_{\rho\in S_3} I_{1\rho(234)}(s_{ij},\tau)$$

$$= \;1 \;+\; (s_{12}^2 + s_{12}s_{23} + s_{23}^2)\,\big[\zeta_2 - 12\,\mathcal{E}_0(4,0;\tau)\big]$$

$$+\; s_{12}s_{13}s_{23}\Big[12\,\mathcal{E}_0(4,0,0;\tau) + 300\,\mathcal{E}_0(6,0,0;\tau) - \frac{5}{2}\,\zeta_3\Big] \;+\; \mathcal{O}(\alpha'^4)$$

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Modular S-transformation $\tau\rightarrow -\frac{1}{\tau}$ follows from $\mathrm{G}_k(-\frac{1}{\tau})=\tau^k\mathrm{G}_k(\tau)$

$$I_{\substack{\text{symm}\\ \text{open}}}(s_{ij},-\tfrac{1}{\tau}) = 1-(s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2)\Big[\frac{T^2}{90}-\frac{2\zeta_2}{3}-\frac{3\zeta_4}{T^2}-\frac{2i\zeta_3}{T}+12\,\mathcal{E}_0(4,0;\tau)+\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)\Big]$$

$$+\, s_{12}s_{13}s_{23}\Big[-\frac{iT^3}{756}+\frac{2i\zeta_2 T}{15}-\frac{\zeta_3}{2}-\frac{35i\zeta_4}{4T}-\frac{12\zeta_2\zeta_3}{T^2}+\frac{15\zeta_5}{2T^2}+\frac{17i\zeta_6}{2T^3}+\frac{72\zeta_2}{T^2}\mathcal{E}_0(4,0,0;\tau)$$

$$+\,300\,\mathcal{E}_0(6,0,0;\tau)+\frac{900i}{T}\mathcal{E}_0(6,0,0,0;\tau)-\frac{900}{T^2}\mathcal{E}_0(6,0,0,0,0;\tau)\Big]\;+\;\mathcal{O}(\alpha'^4)$$

$\Longrightarrow$ coeff's of $q^{0,1,2,\ldots}$ = Laurent polynomials in $T{:=}\pi\tau$ along with MZVs.

III. From closed strings to an elliptic single-valued map

III. 1 Four closed strings on a torus

Four-point closed-string amplitude at one loop (gravitons in type IIA/B)

$$M^{\text{1-loop}}_{\text{closed}}(1,2,3,4) = \left| s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\right|^2 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2}\, J_{\text{closed}}(s_{ij},\tau)$$

$$\underbrace{J_{\text{closed}}(s_{ij},\tau)}_{\text{mod. invariant}} = \left(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j,\tau) \right)\Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $\mathcal{T}(\tau) = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$

- Fourier expansion of the Green function with $z = r + \tau s$ and $r,s \in \mathbb{R}$

$$g(z,\tau) = \frac{\operatorname{Im}\tau}{\pi} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{e^{2\pi i(nr-ms)}}{|m+\tau n|^2} \qquad \text{modular invariant!}$$

- α'-expansion of $J_{\text{closed}}(s_{ij},\tau)$ & generalizations has long history

[Green, Vanhove et al. 1999 - 2019; see Michael Green's talk]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\text{closed}}(s_{ij}, \tau) = \left(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im} \tau} \right) \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j, \tau) \right) \Big|_{z_1=0}$$

Again, Taylor expand the exponentials in s_{ij}

$\Longrightarrow$ need to evaluate $\int_{\mathcal{T}(\tau)} \mathrm{d}^2 z_j$ over $\prod_{i<j} (g(z_{ij}, \tau))^{n_{ij}}$

$\Longrightarrow$ modular invariance order by order in s_{ij}

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\mathrm{closed}}(s_{ij},\tau) \;=\; \Big(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \Big) \; \exp\Big(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j, \tau) \Big) \Big|_{z_1=0}$$

Integrating monomials of Green fct's @ $z = r + \tau s$ over $r, s \in (0,1)$ ...

$$g(z,\tau) \;=\; \frac{\mathrm{Im}\,\tau}{\pi} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{2\pi i (nr - ms)}}{|m+\tau n|^2} \qquad \text{modular invariant!}$$

... naturally lands on non-holomorphic Eisenstein series ...

$$\mathrm{E}_k(\tau) \;=\; \Big(\frac{\mathrm{Im}\,\tau}{\pi}\Big)^k \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{|m+\tau n|^{2k}} \,, \qquad k \geq 2$$

... and generalizations to nested lattice sums "*modular graph functions*".

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\rm closed}(s_{ij},\tau) \ = \ \bigg(\prod_{j=2}^4 \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\, \tau} \bigg) \ \exp\bigg(\sum_{i<j}^4 s_{ij}\, g(z_i - z_j, \tau) \bigg) \bigg|_{z_1=0}$$

Integrating monomials of Green fct's @ $z = r + \tau s$ over $r, s \in (0,1)$ ...

... $\Longrightarrow$ nested lattice sums known as "*modular graph functions*",

e.g. $$\mathrm{E}_k(\tau) \ = \ \Big(\frac{\mathrm{Im}\, \tau}{\pi} \Big)^k \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{|m+\tau n|^{2k}} \, , \qquad k \geq 2\,.$$

First step beyond non-holo' Eisenstein series: double sums $(a,b,c \in \mathbb{N})$

$$C_{a,b,c}(\tau) \ = \ \Big(\frac{\mathrm{Im}\, \tau}{\pi} \Big)^{a+b+c} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \ \sum_{\substack{r,s \in \mathbb{Z} \\ (r,s) \neq (0,0) \\ (r,s) \neq (m,n)}} \frac{1}{|m+\tau n|^{2a}\, |r+\tau s|^{2b}\, |m-r+\tau(n-s)|^{2c}}$$

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\mathrm{closed}}(s_{ij},\tau) \; = \; \bigg(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \bigg) \; \exp\Big(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j, \tau) \Big) \Big|_{z_1=0}$$

E_k and generalizations $\longrightarrow$ (real parts of) iterated Eisenstein integrals

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_{\tau}^{i\infty} \mathrm{d}\tau' \, \mathrm{G}^0_{k_r}(\tau') \, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

[Gangl, Zagier 2000; D'Hoker, Green 1603.00839;

Brödel, OS, Zerbini 1803.00527]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\rm closed}(s_{ij},\tau) \ = \ \bigg(\prod_{j=2}^4 \int_{\mathcal{T}(\tau)} \frac{ {\rm d}^2 z_j}{{\rm Im}\, \tau} \bigg) \ \exp\bigg(\sum_{i<j}^4 s_{ij}\, g(z_i-z_j,\tau) \bigg) \Big|_{z_1=0}$$

E_k and generalizations $\longrightarrow$ (real parts of) iterated Eisenstein integrals

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} {\rm d}\tau' \, {\rm G}^0_{k_r}(\tau')\, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

$\longrightarrow$ series in $q^m \bar q^n$, coefficients are Laurent polynomials in $y := \pi\, {\rm Im}\, \tau$

$$\begin{aligned} J_{\rm closed}(s_{ij},\tau) &= 1 + (s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big[\frac{2y^2}{45} + \frac{2\zeta_3}{y} - 24\, {\rm Re}\, \mathcal{E}_0(4,0;\tau) - \frac{12}{y}\, {\rm Re}\, \mathcal{E}_0(4,0,0;\tau) \Big] \\ &\quad + s_{12}s_{13}s_{23}\Big[-\frac{2y^3}{189} - \zeta_3 - \frac{15\zeta_5}{4y^2} + 600\, {\rm Re}\, \mathcal{E}_0(6,0,0;\tau) \\ &\quad + \frac{900}{y}\, {\rm Re}\, \mathcal{E}_0(6,0,0,0;\tau) + \frac{450}{y^2}\, {\rm Re}\, \mathcal{E}_0(6,0,0,0,0;\tau) \Big] + \mathcal{O}(\alpha'^4) \end{aligned}$$

$\longrightarrow$ structure familiar from open strings & $I^{\rm symm}_{\rm open}(s_{ij}, -\frac{1}{\tau})$ ☺

III. 2 An elliptic single-valued projection?

Compare open- and closed-string α'-expansion

$$I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau}) \quad \longleftrightarrow \quad J_{\rm closed}(s_{ij},\tau)$$

- leading orders $1+\mathcal{O}(\alpha'^2)$ on both sides
- series in q^m or $q^m\bar{q}^n$ & Laurent polynomials in $T := \pi\tau$ or $y := \pi\,{\rm Im}\,\tau$
- compare the first non-trivial order $(\alpha')^2$: very similar coefficients!

$$\begin{aligned} I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau})\,\big|_{(\alpha')^2} &\sim -\frac{T^2}{90}+\frac{2\zeta_2}{3}+\frac{3\zeta_4}{T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau) \\ J_{\rm closed}(s_{ij},\tau)\,\big|_{(\alpha')^2} &\sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}{\rm Re}\,\mathcal{E}_0(4,0,0;\tau) \end{aligned}$$

$\rightarrow$ no closed-string analogue of ζ_{2k}, only $\mathbf{sv}$(MZV) ζ_3 survives

$\rightarrow$ closed string: real parts ${\rm Re}\,\mathcal{E}_0(\ldots)$ of iterated Eisenstein int's

III. 2 An elliptic single-valued projection?

How to map open-string data to closed-string data?

$$I_{\substack{\text{symm}\\ \text{open}}}(s_{ij}, -\tfrac{1}{\tau})\,\Big|_{(\alpha')^2} \sim -\frac{T^2}{90} + \frac{2\zeta_2}{3} + \frac{3\zeta_4}{T^2} + \frac{2i\zeta_3}{T} - 12\,\mathcal{E}_0(4,0;\tau) - \frac{12i}{T}\,\mathcal{E}_0(4,0,0;\tau)$$

$$J_{\text{closed}}(s_{ij}, \tau)\,\Big|_{(\alpha')^2} \sim \frac{2y^2}{45} + \frac{2\zeta_3}{y} - 24\,\mathrm{Re}\,\mathcal{E}_0(4,0;\tau) - \frac{12}{y}\,\mathrm{Re}\,\mathcal{E}_0(4,0,0;\tau)$$

III. 2 An elliptic single-valued projection?

How to map open-string data to closed-string data?

$$I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau})\,\Big|_{(\alpha')^2} \sim -\frac{T^2}{90}+\frac{2\zeta_2}{3}+\frac{3\zeta_4}{T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\,\mathcal{E}_0(4,0,0;\tau)$$

$$J_{\rm closed}(s_{ij},\tau)\,\Big|_{(\alpha')^2} \sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}\,{\rm Re}\,\mathcal{E}_0(4,0,0;\tau)$$

Engineer <u>*elliptic single-valued projection*</u> **esv**

$$\textbf{esv} \ : \ \begin{cases} \text{(i)} : & \zeta_{n_1,n_2,\ldots} \ \rightarrow \ \textbf{sv}(\zeta_{n_1,n_2,\ldots}) \\[2ex] \text{(ii)} : & T \ \rightarrow \ 2iy \ \ \text{i.e.} \ \ \tau \ \rightarrow \ 2i\,{\rm Im}\,\tau \\[2ex] \text{(iii)} : & \mathcal{E}_0(k_1,\ldots;\tau) \ \rightarrow \ 2\,{\rm Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

to match above expressions @ $(\alpha')^2$ and in fact complete $(\alpha')^{\leq 6}$ orders!

$$\boxed{\textbf{esv}\ I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau}) \ = \ J_{\rm closed}(s_{ij},\tau)}$$

[Brödel, OS, Zerbini 1803.00527]

III. 2 An elliptic single-valued projection?

Conjectural *elliptic single-valued projection* **esv** (works to order α'^6)

$$\mathbf{esv}\ :\ \begin{cases} \text{(i)}: & \zeta_{n_1,n_2,\ldots} \to \mathbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ \text{(ii)}: & T \to 2iy \ \text{ i.e. } \ \tau \to 2i\,\mathrm{Im}\,\tau \\ \text{(iii)}: & \mathcal{E}_0(k_1,\ldots;\tau) \to 2\,\mathrm{Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

$$\boxed{\mathbf{esv}\, I_{\substack{\text{symm}\\ \text{open}}}(s_{ij},-\tfrac{1}{\tau}) \ = \ J_{\text{closed}}(s_{ij},\tau)}$$

conjectured in **[Brödel, OS, Zerbini 1803.00527]**

- so far requires ad-hoc convention how to use shuffle multiplication of $\mathcal{E}_0$
- **esv** should relate to equivariant iterated Eisenstein integrals of Brown **[Brown 1407.5167, 1707.01230, 1708.03354]**
- resonates with $J_{\text{closed}}(s_{ij},\tau) \Rightarrow$ infinite sums of **sv**(polylogarithms) **[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]**

III. 3 Modular graph forms from heterotic strings

Loose end: What is **esv** $I_{1234}(s_{ij},-\frac{1}{\tau})$ *without* symmetrizing $\int_{z_2<z_3<z_4}$?

$\longrightarrow$ more general integral over tori from heterotic strings at one loop

$$J_{\text{het}}(s_{ij},\tau) \ = \ \left(\prod_{j=1}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) V_2(z_1,z_2,z_3,z_4|\tau) \ \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_{ij},\tau) \right)$$

Elliptic fct. $V_2(z_1,z_2,z_3,z_4|\tau)$ of modular weight $(2,0)$ defined by

cyclic product of Kronecker–Eisenstein series $F(z,\beta,\tau) = \frac{\theta_1'(0,\tau)\theta_1(z+\beta,\tau)}{\theta_1(z,\tau)\theta_1(\beta,\tau)}$

$$V_2(z_1,z_2,z_3,z_4|\tau) \ = \ F(z_{12},\beta,\tau)\, F(z_{23},\beta,\tau)\, F(z_{34},\beta,\tau)\, F(z_{41},\beta,\tau)\, \Big|_{\beta^{-2}} .$$

[Dolan, Goddard 0710.3743]

α'-expansion of $J_{\text{het}}(s_{ij},\tau) \ \Rightarrow$ "modular graph forms" of weight $(2,0)$

[D'Hoker, Green 1603.00839]

[Gerken, Kleinschmidt, OS 1811.02548]

III. 3 Modular graph forms from heterotic strings

Modular graph forms of weight $(2,0)$ from $\frac{\partial}{\partial\tau}$(modular graph functions)

$$
\begin{aligned}
J_{\rm het}(s_{ij},\tau) \ &= \ \Big(\prod_{j=1}^4 \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \Big) \ V_2(z_1,z_2,z_3,z_4|\tau) \ \exp\Big(\sum_{i<j} s_{ij}\, g(z_{ij},\tau) \Big) \\
&= 2\pi i \Big\{ 3 s_{13} \, \frac{\partial \mathrm{E}_2}{\partial \tau} \ + \ \frac{2}{3} (s_{13}^2 + 2 s_{12} s_{23}) \, \frac{\partial \mathrm{E}_3}{\partial \tau} \Big\} \ + \ \mathcal{O}(\alpha'^3)
\end{aligned}
$$

Expansion around the cusp resembles modular graph functions ($y = \pi \mathrm{Im}\,\tau$)

$$
\begin{aligned}
J_{\rm het}(s_{ij},\tau) \ &= \ \pi^2 \Big\{ s_{13} \Big(\frac{2y}{15} - \frac{3\zeta_3}{y^2} \Big) + (s_{13}^2 + 2 s_{12} s_{23}) \Big(\frac{4y^2}{945} - \frac{\zeta_5}{y^3} \Big) \\
&+ \ s_{13}(s_{13}^2 - s_{12}s_{23}) \Big(\frac{4y^3}{945} + \frac{2\zeta_3}{5} - \frac{5\zeta_5}{y^2} - \frac{3\zeta_7}{2y^4} \Big) \ + \ \mathcal{O}(\alpha'^4) \Big\} \ + \ \mathcal{O}(q,\bar q)
\end{aligned}
$$

Suppressed terms $\mathcal{O}(q,\bar q)$ expressible via known $\mathcal{E}_0(\ldots)$ and $\overline{\mathcal{E}_0(\ldots)}$.

[Gerken, Kleinschmidt, OS 1811.02548]

III. 3 Modular graph forms from heterotic strings

Compare heterotic-string integral ...

$$J_{\rm het}(s_{ij},\tau) \ = \ \pi^2 \Big\{ s_{13}\Big(\frac{2y}{15} - \frac{3\zeta_3}{y^2}\Big) + (s_{13}^2 + 2s_{12}s_{23})\Big(\frac{4y^2}{945} - \frac{\zeta_5}{y^3}\Big)$$
$$+ \ s_{13}(s_{13}^2 - s_{12}s_{23})\Big(\frac{4y^3}{945} + \frac{2\zeta_3}{5} - \frac{5\zeta_5}{y^2} - \frac{3\zeta_7}{2y^4}\Big) \ + \ \mathcal{O}(\alpha'^4)\Big\} \ + \ \mathcal{O}(q,\bar q)$$

... with open-string integrals (with $\mathcal{O}(q)$ referring to known $\mathcal{E}_0(\ldots)$)

$$\frac{2}{3} I_{1234}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3} I_{1324}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3} I_{1243}(s_{ij},-\tfrac{1}{\tau}) \ = \ s_{13}\Big(\frac{iT}{60} - \frac{i\zeta_2}{2T} - \frac{3\zeta_3}{2T^2} + \frac{3i\zeta_4}{2T^3}\Big)$$
$$+ (s_{13}^2 + 2s_{12}s_{23})\Big(\frac{T^2}{3780} - \frac{\zeta_2}{36} + \frac{\zeta_4}{4T^2} - \frac{i\zeta_5}{T^3} - \frac{5\zeta_6}{4T^4}\Big)$$
$$+ s_{13}(s_{13}^2 - s_{12}s_{23})\Big(-\frac{iT^3}{7560} + \frac{i\zeta_2 T}{90} - \frac{\zeta_3}{20} - \frac{3i\zeta_4}{4T} - \frac{5\zeta_5}{2T^2} + \frac{\zeta_2\zeta_3}{2T^2}$$
$$+ \frac{29i\zeta_6}{12T^3} + \frac{3\zeta_3\zeta_4}{2T^4} + \frac{3\zeta_7}{T^4} - \frac{21i\zeta_8}{4T^5}\Big) + \mathcal{O}(\alpha'^4) + \mathcal{O}(q)$$

Combinations of orderings $I_{1ijk}(s_{ij},-\frac{1}{\tau})$ vanishes upon symmetrization.

III. 3 Modular graph forms from heterotic strings

Compare heterotic-string integral ...

$$J_{\rm het}(s_{ij},\tau) \ = \ \pi^2 \Big\{ s_{13}\Big(\frac{2y}{15} - \frac{3\zeta_3}{y^2}\Big) + (s_{13}^2 + 2s_{12}s_{23})\Big(\frac{4y^2}{945} - \frac{\zeta_5}{y^3}\Big)$$

$$+ \ s_{13}(s_{13}^2 - s_{12}s_{23})\Big(\frac{4y^3}{945} + \frac{2\zeta_3}{5} - \frac{5\zeta_5}{y^2} - \frac{3\zeta_7}{2y^4}\Big) \ + \ \mathcal{O}(\alpha'^4)\Big\} \ + \ \mathcal{O}(q,\bar q)$$

... with open-string integrals (with $\mathcal{O}(q)$ referring to known $\mathcal{E}_0(\ldots)$)

$$\frac{2}{3}I_{1234}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1324}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1243}(s_{ij},-\tfrac{1}{\tau}) \ = \ s_{13}\Big(\frac{iT}{60} - \frac{i\zeta_2}{2T} - \frac{3\zeta_3}{2T^2} + \frac{3i\zeta_4}{2T^3}\Big)$$

$$+ (s_{13}^2 + 2s_{12}s_{23})\Big(\frac{T^2}{3780} - \frac{\zeta_2}{36} + \frac{\zeta_4}{4T^2} - \frac{i\zeta_5}{T^3} - \frac{5\zeta_6}{4T^4}\Big) + \mathcal{O}(\alpha'^3) + \mathcal{O}(q)$$

Tests up to α'^3 motivate conjecture (also reproducing $\bar q^{N>0}q^0$ terms)

$$\boxed{J_{\rm het}(s_{ij},\tau) \ = \ (2\pi i)^2\, \mathbf{esv}\Big(\frac{2}{3}I_{1234}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1324}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1243}(s_{ij},-\tfrac{1}{\tau})\Big) + \mathcal{O}(q)}$$

[Gerken, Kleinschmidt, OS 1811.02548]

IV. Conclusion

• α'-expansion of string amplitudes $\longleftrightarrow$ periods of moduli spaces $\mathcal{M}_{g;n}$

	open strings	closed strings
tree	disk $\Rightarrow$ multiple zeta values	sphere $\Rightarrow$ single-valued MZVs
level	(MZVs) $=$ polylog's at $z=1$	$=$ single-valued polylog's at $z=1$
one	cylinder / Möbius strip	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ $\Rightarrow$ modular graph fct's
loop	$\Rightarrow$ elliptic MZVs	(modular invariant fct's of τ)

IV. Conclusion

- α'-expansion of string amplitudes $\longleftrightarrow$ periods of moduli spaces $\mathcal{M}_{g;n}$

	open strings		closed strings
tree level	disk $\Rightarrow$ multiple zeta values (MZVs) = polylog's at $z=1$	$\xrightarrow{\textbf{sv}}$	sphere $\Rightarrow$ single-valued MZVs = single-valued polylog's at $z=1$
one loop	cylinder / Möbius strip $\Rightarrow$ elliptic MZVs	$\xrightarrow{\textbf{esv}}$	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ $\Rightarrow$ modular graph fct's (modular invariant fct's of τ)

IV. Conclusion

- α'-expansion of string amplitudes $\longleftrightarrow$ periods of moduli spaces $\mathcal{M}_{g;n}$

- conjectural elliptic single-valued projection from one-loop α'-expansions:

  esv : eMZVs (open string) $\rightarrow$ modular graph forms (closed string)

- broader picture: complex integrals "$\mathrm{d}^2 z$" = **sv**(contour integrals "$\mathrm{d}z$")

  [Schnetz 1302.6445 & Brown, Dupont 1810.07682]

- also expect relations higher-genus modular graph fct's $\leftrightarrow$ open strings

  [D'Hoker, Green, Pioline 1712.06135, 1806.02691]

Thank you for your attention !