Feynman integrals associated to elliptic curves

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Part I: No elliptic curves
Part II: One elliptic curve
Part III: Several elliptic curves
Part I

No elliptic curves

(Introduction to Feynman integrals)
Experiments in high-energy physics

Collision Event with 2 Jets

ATLAS EXPERIMENT
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For a theoretical description we need to know the scattering amplitude:

\[ p_1, \ldots, p_N \text{ext.} \]

Momentum conservation: \( p_1 + \ldots + p_{N\text{ext}} = 0. \)
We may compute the scattering amplitude within perturbation theory:

\[ \begin{align*}
\frac{p_1}{p_2} + & \frac{p_3}{p_4} + \cdots \quad \left\{ O(g^2) \right\} \\
\frac{p_1}{p_2} & + \frac{p_3}{p_4} + \cdots \quad \left\{ O(g^4) \right\} \\
\frac{p_1}{p_2} & + \frac{p_3}{p_4} + \cdots \quad \left\{ O(g^6) \right\}
\end{align*} \]
Feynman integrals

Associate to a Feynman graph $G$ with $N_{\text{ext}}$ external lines, $n$ internal lines and $l$ loops the set of Feynman integrals

$$I_{\nu_1\nu_2...\nu_n} = \int \frac{d^Dk_1}{(2\pi)^D} \cdots \frac{d^Dk_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{(q_j^2 - m_j^2)^{\nu_j}},$$

with $\nu_j \in \mathbb{Z}$ and $\nu = \nu_1 + \ldots + \nu_n$. 
If for some exponent we have $v_j = 0$, the corresponding propagator is absent and the topology simplifies:
Integration by parts

Within dimensional regularisation we have for any loop momentum \( k_i \) and \( v \in \{ p_1, \ldots, p_{N_{\text{ext}}}, k_1, \ldots, k_l \} \)

\[
\int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} \, \nu^\mu \prod_{j=1}^n \frac{1}{(q_j^2 - m_j^2)^{\nu_j}} = 0.
\]

Working out the derivatives leads to relations among integrals with different sets of indices \((\nu_1, \ldots, \nu_n)\).

This allows us to express most of the integrals in terms of a few master integrals.

Tkachov '81, Chetyrkin '81
Expressing all integrals in terms of the master integrals requires to solve a rather large linear system of equations.

This system has a block-triangular structure, originating from subtopologies.

Order the integrals by complexity (more propagators ⇒ more difficult)

Solve the system bottom-up, re-using the results for the already solved sectors.

Laporta ’01
Let $x_k$ be a kinematic variable. Let $I_i \in \{I_1, \ldots, I_{N_{\text{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_{\text{master}}} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)
Let us formalise this:

\[ \vec{I} = (I_1, \ldots, I_{N_{\text{master}}}) , \quad \text{set of master integrals}, \]
\[ \vec{x} = (x_1, \ldots, x_{N_B}) , \quad \text{set of kinematic variables} \text{ the master integrals depend on}. \]

We obtain a system of differential equations of Fuchsian type

\[ d\vec{I} + A \vec{I} = 0, \]

where \( A \) is a matrix-valued one-form

\[ A = \sum_{i=1}^{N_B} A_i dx_i. \]

The matrix-valued one-form \( A \) satisfies the integrability condition

\[ dA + A \wedge A = 0 \quad (\text{flat Gauß-Manin connection}). \]

Computation of Feynman integrals reduced to solving differential equations!
If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$(d + A')\vec{J} = 0, \quad A' = UAU^{-1} + UdU^{-1}$$

Suppose one finds a transformation matrix $U$, such that

$$A' = \epsilon \sum_j C_j \, d \ln p_j(\vec{x}),$$

where

- $\epsilon$ appears only as prefactor,
- $C_j$ are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn '13
Transformation to the $\varepsilon$-form

We may

- change the basis of the master integrals

$$\vec{I} \rightarrow U\vec{I},$$

where $U$ is rational in the kinematic variables

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, \ldots, x_{N_B}) \rightarrow (x'_1, \ldots, x'_{N_B}),$$

often done to absorb square roots.

Becchetti, Bonciani, '17, Besier, van Straten, S.W., '18
Multiple polylogarithms

Definition based on nested sums:

\[
\text{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}
\]

Definition based on iterated integrals:

\[
G(z_1,\ldots,z_k;y) = \int_0^y dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{k-1}} dt_k \frac{1}{t_1-z_1} \frac{1}{t_2-z_2} \ldots \frac{1}{t_k-z_k}
\]

Conversion:

\[
\text{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1x_2}, \ldots, \frac{1}{x_1\ldots x_k}; 1\right)
\]

Short hand notation:

\[
G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = G(0,\ldots,0,\underbrace{z_1,\ldots,z_{k-1},0,\ldots,0}_{m_1-1},z_k;y)
\]
Let us consider a simple example: One integral $I$ in one variable $x$ with boundary condition $I(0) = 1$. Consider the differential equation

$$(d + A)I = 0, \quad A = -\varepsilon d \ln (x - 1).$$

Note that

$$d \ln (x - 1) = \frac{dx}{x - 1}$$

and

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \ldots$$
Iterated integrals

For $\omega_1, \ldots, \omega_k$ differential 1-forms on a manifold $M$ and $\gamma : [0, 1] \to M$ a path, write for the pull-back of $\omega_j$ to the interval $[0, 1]$

$$f_j(\lambda) \, d\lambda = \gamma^* \omega_j.$$  

The iterated integral is defined by (Chen ’77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \cdots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$
Iterated integrals

Example 1: Multiple polylogarithms (Goncharov '98)

\[ \omega_j = \frac{d\lambda}{\lambda - z_j}. \]

Example 2: Iterated integrals of modular forms (Brown '14): \( f_j(\tau) \) a modular form,

\[ \omega_j = 2\pi i f_j(\tau) \, d\tau. \]

Example 3: Iterated integrals on a covering space of a fixed single elliptic curve, also known as “elliptic polylogarithms”
(Broedel, Duhr, Dulat, Penante, Tancredi, '17-'19).
Part II

One elliptic curve

(Feynman integrals beyond multiple polylogarithms)
Single-scale Feynman integrals beyond multiple polylogarithms

Not all Feynman integrals are expressible in terms of multiple polylogarithms!
The Picard-Fuchs operator

Let $I$ be one of the master integrals $\{I_1, \ldots, I_{N_{\text{master}}}\}$. Choose a path $\gamma : [0, 1] \to M$ and study the integral $I$ as a function of the path parameter $\lambda$.

Instead of a system of $N_{\text{master}}$ first-order differential equations

$$(d + A)\vec{I} = 0,$$

we may equivalently study a single differential equation of order $N_{\text{master}}$

$$\sum_{j=0}^{N_{\text{master}}} p_j(\lambda) \frac{d^j}{d\lambda^j} I = 0.$$ 

We may work modulo sub-topologies and $\varepsilon$-corrections:

$$L = \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} : L I = 0 \mod (\text{sub-topologies, } \varepsilon\text{-corrections})$$
Suppose the differential operator factorises into linear factors:

\[
L = \left( a_r(\lambda) \frac{d}{d\lambda} + b_r(\lambda) \right) \ldots \left( a_2(\lambda) \frac{d}{d\lambda} + b_2(\lambda) \right) \left( a_1(\lambda) \frac{d}{d\lambda} + b_1(\lambda) \right)
\]

Iterated first-order differential equation.

Denote homogeneous solution of the \( j \)-th factor by

\[
\psi_j(\lambda) = \exp \left( - \int_0^\lambda d\kappa \frac{b_j(\kappa)}{a_j(\kappa)} \right).
\]

Full solution given by iterated integrals

\[
C_1 \psi_1(\lambda) + C_2 \psi_1(\lambda) \int_0^\lambda d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1) \psi_1(\lambda_1)} + C_3 \psi_1(\lambda) \int_0^\lambda d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1) \psi_1(\lambda_1)} \int_0^{\lambda_1} d\lambda_2 \frac{\psi_3(\lambda_2)}{a_2(\lambda_2) \psi_2(\lambda_2)} + \ldots
\]

Multiple polylogarithms are of this form.
Suppose the differential operator

\[
\sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j}
\]

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

\[
a_j(\lambda) \frac{d^2}{d\lambda^2} + b_j(\lambda) \frac{d}{d\lambda} + c_j(\lambda)
\]
The differential operator of the second-order differential equation

\[
\left[ k (1 - k^2) \frac{d^2}{dk^2} + (1 - 3k^2) \frac{d}{dk} - k \right] f(k) = 0
\]

is irreducible.

The solutions of the differential equation are \( K(k) \) and \( K(\sqrt{1 - k^2}) \), where \( K(k) \) is the complete elliptic integral of the first kind:

\[
K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.
\]
An example from physics: The two-loop sunrise integral

\[ S_{\nu_1\nu_2\nu_3}(D,x) = \]

Picard-Fuchs operator for \( S_{111}(2,x) \):

\[ L = x(x - 1)(x - 9) \frac{d^2}{dx^2} + (3x^2 - 20x + 9) \frac{d}{dx} + (x - 3) \]

(Broadhurst, Fleischer, Tarasov '93)

Irreducible second-order differential operator.

Picard-Fuchs operator for the periods of a family of elliptic curves.
The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:
  \[-x_1x_2x_3x + (x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1) = 0\]

- From the maximal cut:
  \[v^2 - (u - x)(u - x + 4)(u^2 + 2u + 1 - 4x) = 0\]

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods $\psi_1$, $\psi_2$ of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16
Recall

\[ x = \frac{p^2}{m^2}. \]

Set

\[ \tau = \frac{\psi_2}{\psi_1}, \quad q = e^{2i\pi \tau}. \]

Change variable from \( x \) to \( \tau \) (or \( q \)) (Bloch, Vanhove, '13).
The periods $\psi_1$ and $\psi_2$ generate a lattice. Any other basis as good as $(\psi_2, \psi_1)$. Convention: Normalise $(\psi_2, \psi_1) \rightarrow (\tau, 1)$ where $\tau = \psi_2/\psi_1$.

Change of basis:

$$
\begin{pmatrix}
\psi'_2 \\
\psi'_1
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix},
$$

Transformation should be invertible:

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \text{SL}(2, \mathbb{Z}),
$$

In terms of $\tau$ and $\tau'$:

$$
\tau' = \frac{a\tau + b}{c\tau + d}. 
$$
The $\varepsilon$-form of the differential equation for the sunrise

It is not possible to obtain an $\varepsilon$-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression $\psi_1/\pi$ from the master integrals in the sunrise sector one obtains an $\varepsilon$-form:

$$I_1 = 4\varepsilon^2 S_{110} (2 - 2\varepsilon, x), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111} (2 - 2\varepsilon, x), \quad I_3 = \frac{1}{\varepsilon} \frac{1}{2\pi i} \frac{d}{d\tau} I_2 + \frac{1}{24} \left( 3x^2 - 10x - 9 \right) \frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from $x$ to $\tau$, one obtains

$$\frac{d}{d\tau} \vec{I} = \varepsilon A(\tau) \vec{I},$$

where $A(\tau)$ is an $\varepsilon$-independent $3 \times 3$-matrix whose entries are modular forms.
The $\varepsilon$-form of the differential equation for the sunrise

The matrix $A(\tau)$ is given by

$$A(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2(\tau) & 0 \\ \frac{1}{4}f_3(\tau) & f_4(\tau) & -f_2(\tau) \end{pmatrix},$$

where $f_2$, $f_3$ and $f_4$ are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively.

$I_1$, $I_2$ and $I_3$ are expressed as iterated integrals of modular forms to all orders in $\varepsilon$.

Adams, S.W., '17, '18
Feynman integrals evaluating to iterated integrals of modular forms

This applies to a wider class of Feynman integrals:
Several elliptic curves

(An example from top-pair production)
\[ I_{v_1v_2v_3v_4v_5v_6v_7}(D, \frac{s}{m^2}, \frac{t}{m^2}) = (m^2)^7 \sum_{j=1}^{7} \frac{1}{P_j} \]

\[ \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \prod_{j=1}^{7} \frac{1}{P_j}, \]

\[ p_1^2 = p_2^2 = 0, \quad p_3^2 = p_4^2 = m^2, \]

\[ s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2. \]
Sunrise integral: An elliptic curve can be obtained either from

- Feynman graph polynomial
- maximal cut

The periods $\psi_1, \psi_2$ are the solutions of the homogeneous differential equations.

Adams, Bogner, S.W., '13, '14

In general: The maximal cuts are solutions of the homogeneous differential equations.

Primo, Tancredi, '16

Search for Feynman integrals, whose maximal cuts are periods of an elliptic curve.
Maximal cuts

Maximal cut: For a Feynman integral

\[ I_{\nu_1\nu_2...\nu_n} = \left( \mu^2 \right)^{\nu - lD/2} \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}} \]

take the \( n \)-fold residue at

\[ P_1 = \ldots = P_n = 0 \]

of the integrand and integrate over the remaining \((lD - n)\) variables along a contour \( C \).
Maximal cuts

Sunrise:

\[
\text{MaxCut}_C I_{1001001} (2 - 2\varepsilon) = \\
\frac{um^2}{\pi^2} \int_C \frac{dP}{(P-t)^{\frac{1}{2}} (P-t+4m^2)^{\frac{1}{2}} (P^2 + 2m^2P - 4m^2t + m^4)^{\frac{1}{2}}} + O(\varepsilon).
\]

Double box:

\[
\text{MaxCut}_C I_{1111111} (4 - 2\varepsilon) = \\
\frac{um^6}{4\pi^4s^2} \int_C \frac{dP}{(P-t)^{\frac{1}{2}} (P-t+4m^2)^{\frac{1}{2}} \left( P^2 + 2m^2P - 4m^2t + m^4 - \frac{4m^2(m^2-t)^2}{s} \right)^{\frac{1}{2}}} + O(\varepsilon).
\]
Three elliptic curves

\[ E^{(a)} : \quad w^2 = (z - t)(z - t + 4m^2) \left( z^2 + 2m^2z - 4m^2t + m^4 \right) \]

\[ E^{(b)} : \quad w^2 = (z - t)(z - t + 4m^2) \left( z^2 + 2m^2z - 4m^2t + m^4 - \frac{4m^2 (m^2 - t)^2}{s} \right) \]

\[ E^{(c)} : \quad w^2 = (z - t)(z - t + 4m^2) \left( z^2 + \frac{2m^2(s + 4t)}{(s - 4m^2)}z + \frac{sm^2(m^2 - 4t) - 4m^2t^2}{s - 4m^2} \right) \]
Remarks

• \( E^{(a)} \) gives rise to iterated integrals of modular forms of \( \Gamma_1(6) \).

• For \( s \to \infty \) the curves \( E^{(b)} \) and \( E^{(c)} \) degenerate to \( E^{(a)} \).

• If we would have only one curve, we expect that the result can be written in elliptic polylogarithms.

• We have three elliptic curves.
The differential equation for the master integrals can be brought into the form

\[ d\vec{I} = \varepsilon A \vec{I}, \]

where \( A \) is independent of \( \varepsilon \).

The Laurent expansion in \( \varepsilon \) of all master integrals can be computed systematically to all orders in \( \varepsilon \) in terms of iterated integrals.

The solution

- reduces to multiple polylogarithms for \( t = m^2 \) and

- reduces to iterated integrals of modular forms of \( \Gamma_1(6) \) for \( s = \infty \).

Adams, Chaubey, S.W., '18
Conclusions

• Loop integrals with **masses important** for top, $W/Z$- and $H$-physics.

• May involve **elliptic sectors** from two loops onwards.

• There is a class of Feynman integrals evaluating to **iterated integrals of modular forms**.

• The planar double box integral relevant to $t\bar{t}$-production with a closed top loop depends on **two variables** and involves **several elliptic** sub-sectors. More than one elliptic curve occurs. Results expressed in terms of Chen’s iterated integrals.

• We may expect more results in the near future.